

The following note is useful for this discussion: [Note 18](#).

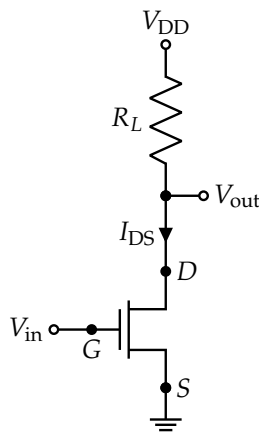
1. Using a Nonlinear NMOS Transistor for Amplification

Consider the following schematic where $V_{DD} = 1.5\text{ V}$, $R_L = 400\ \Omega$ and the NMOS transistor has threshold voltage $V_{th} = 0.2\text{ V}$. We are interested in analyzing the response of this circuit to input voltages of the form $V_{in}(t) = V_{in,DC} + v_{in,AC}(t)$, where $V_{in,DC}$ is some constant voltage and $v_{in,AC}(t) = 0.001 \cos(\omega t)\text{ V}$ is a sinusoidal signal whose magnitude is much smaller than $V_{in,DC}$.

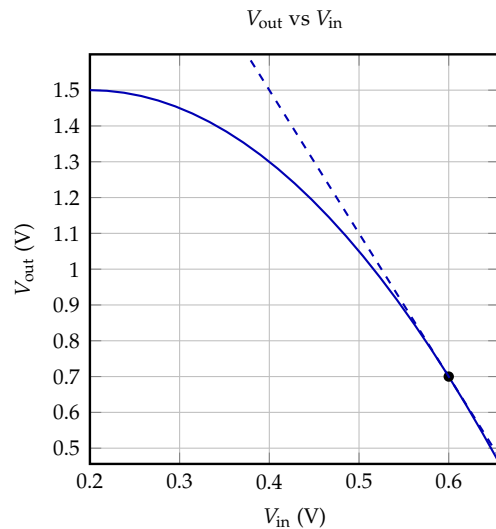
The I-V relationship of an NMOS can be modeled as non-linear functions over different regions of operation. For simplicity, let's just focus on the case when $0 \leq V_{GS} - V_{th} < V_{DS}$. In this regime of interest, the relevant I-V relationship is given by

$$I_{DS}(V_{GS}) = \frac{K}{2}(V_{GS} - V_{th})^2 \tag{1}$$

where K is a constant that depends on the NMOS transistor size and properties.



(a) NMOS Transistor circuit



(b) V_{out} vs V_{in} in the regime of interest. Tangent is drawn at the operating point $V_{in,DC} = 0.6\text{ V}$, $V_{out,DC} = 0.7\text{ V}$

Figure 1: NMOS figures.

From Ohm's law and KCL, we know that

$$V_{out}(t) = V_{DD} - R_L I_{DS}(t). \tag{2}$$

Note from Figure 1a that $V_{in} = V_{GS}$ and $V_{out} = V_{DS}$. In Figure 1b, we can see the curve of V_{out} vs V_{in} in the transistor operating regime of interest.

- (a) Using eq. (1) and eq. (2), express $V_{out}(t)$ as a function of $V_{in}(t)$ symbolically. (You can use V_{DD} , R_L , V_{in} , K , V_{th} in your answer.)

Solution:

$$V_{\text{out}}(t) = V_{\text{DD}} - R_L I_{\text{DS}}(t) \Big|_{V_{\text{GS}}=V_{\text{in}}(t)} \quad (3)$$

$$= V_{\text{DD}} - R_L \frac{K}{2} (V_{\text{in}}(t) - V_{\text{th}})^2 \quad (4)$$

A plot of eq. (4) is shown in Figure 1b.

- (b) We can decompose the input into constant (i.e., DC) and time-varying (i.e., AC) components to obtain $V_{\text{in}}(t) = V_{\text{in,DC}} + v_{\text{in,AC}}(t)$. **Linearize $V_{\text{out}}(t)$ as a function of $V_{\text{in}}(t)$ about $V_{\text{in}} = V_{\text{in,DC}}$. What is the slope of the linearized function?**

Solution: Using eq. (4), we have

$$\widehat{V_{\text{out}}}(V_{\text{in}}; V_{\text{in,DC}}) = V_{\text{DD}} - R_L \frac{K}{2} (V_{\text{in,DC}} - V_{\text{th}})^2 + \frac{dV_{\text{out}}}{dV_{\text{in}}}(V_{\text{in}} - V_{\text{in,DC}}) \quad (5)$$

where

$$\frac{dV_{\text{out}}}{dV_{\text{in}}} = -R_L K (V_{\text{in,DC}} - V_{\text{th}}) \quad (6)$$

This is our slope. We can rewrite this as $-R_L g_m$ where $g_m := K(V_{\text{in,DC}} - V_{\text{th}})$ is defined to be the *transconductance gain*. Altogether, we have

$$\widehat{V_{\text{out}}}(V_{\text{in}}; V_{\text{in,DC}}) = V_{\text{DD}} - R_L \frac{K}{2} (V_{\text{in,DC}} - V_{\text{th}})^2 - R_L g_m v_{\text{in,AC}} \quad (7)$$

- (c) Next, we can also decompose the output V_{out} into DC and AC components to obtain $V_{\text{out}} = V_{\text{out,DC}} + v_{\text{out,AC}}(t)$. **What is $V_{\text{out,DC}}$ from the linearized representation in part 1.b? Simplify the linear approximation to be in terms of $v_{\text{out,AC}}(t)$ and $v_{\text{in,AC}}(t)$, for very small $v_{\text{in,AC}}(t)$.**

Solution: The linearized equation is

$$\widehat{V_{\text{out}}}(V_{\text{in}}; V_{\text{in,DC}}) = V_{\text{DD}} - R_L \frac{K}{2} (V_{\text{in,DC}} - V_{\text{th}})^2 - R_L g_m v_{\text{in,AC}} \quad (8)$$

Note that the term $-R_L g_m v_{\text{in,AC}}$ is time-varying due to the $v_{\text{in,AC}}$ term being time-varying. Hence, we may define $V_{\text{out,DC}} := V_{\text{DD}} - R_L \frac{K}{2} (V_{\text{in,DC}} - V_{\text{th}})^2$.

For very small $v_{\text{in,AC}}(t)$, we may write

$$V_{\text{out}} \approx V_{\text{out,DC}} - R_L g_m v_{\text{in,AC}} \quad (9)$$

since the linear approximation is close to the true value of V_{out} . Simplifying this, we obtain

$$v_{\text{out,AC}}(t) \approx -R_L g_m v_{\text{in,AC}}(t) \quad (10)$$

- (d) **For very small $v_{\text{in,AC}}(t)$, what circuit element can we use to represent the I-V relation between $v_{\text{out,AC}}(t)$ and $\Delta I_{\text{DS}} := g_m v_{\text{in,AC}}(t)$? Draw this circuit element.**

Solution: To find the I-V relationship, we compute

$$\frac{v_{\text{out,AC}}(t)}{\Delta I_{\text{DS}}} = \frac{-R_L g_m v_{\text{in,AC}}(t)}{g_m v_{\text{in,AC}}(t)} = -R_L \quad (11)$$

This relationship is not exactly the I-V relationship of a resistor (due to the negative sign), so we can use a voltage-controlled current source as shown in Figure 2.

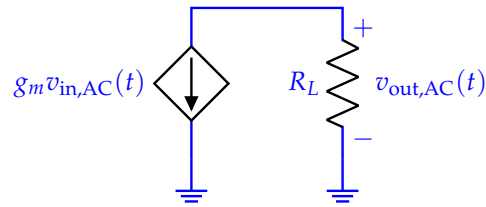
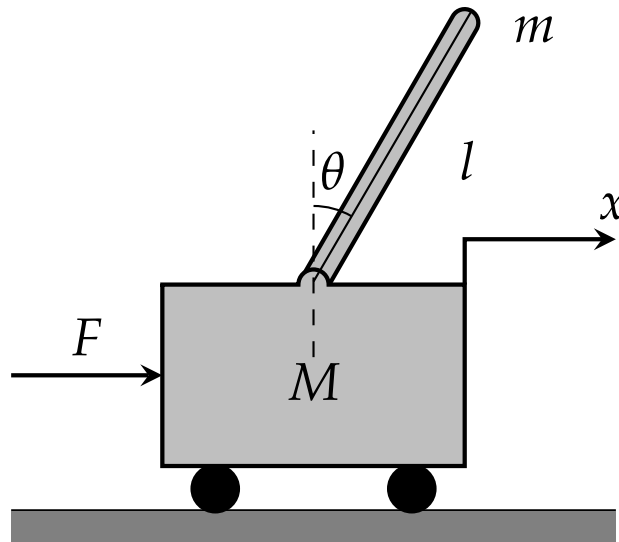


Figure 2: Small signal model for NMOS circuit in Figure 1a.

2. Feedback Control and Linearization

Consider the problem of balancing a pole on a cart as follows:



The mass of the cart itself is M , the length of the rod is l , and the mass of the rod is m . The angle θ is measured with respect to the vertical as shown above, and x is the horizontal translation of the cart (i.e., along the x -axis). The force F is the control input to the system. Assume that all of the mass on the rod is concentrated at the very end of the rod. Further assume that there is no friction. The following differential equations are derived from the physics describing the cart-pole system:

$$\frac{d^2x(t)}{dt^2} = \frac{1}{M + m \sin^2(\theta)} \left(F + m \sin(\theta) \left(l \left(\frac{d\theta}{dt} \right)^2 - g \cos(\theta) \right) \right) \quad (12)$$

$$\frac{d^2\theta(t)}{dt^2} = \frac{1}{l(M + m \sin^2(\theta))} \left(-F \cos(\theta) - ml \left(\frac{d\theta}{dt} \right)^2 \cos(\theta) \sin(\theta) + (M + m)g \sin(\theta) \right) \quad (13)$$

Deriving these equations is out of scope. The task is as follows: we would like the pole to remain upright and the cart to be at the origin (i.e., $x = 0$). The cart and pole must also be stationary. In this problem, we will use Jacobian linearization and state feedback to derive a controller that can achieve this goal for us.

- (a) Show that an appropriate state space would be $\vec{x}(t) = \begin{bmatrix} x \\ \frac{dx}{dt} \\ \theta \\ \frac{d\theta}{dt} \end{bmatrix}$. Find an appropriate control

input $u(t)$, and then find $\vec{f}(\vec{x}, u)$ such that $\frac{d}{dt}\vec{x} = \vec{f}(\vec{x}, u)$. Assume $m = M = l = 1$.

Solution: Since we have second derivatives in translational displacement and angle of the pole, we can define our state as

$$\vec{x}(t) = \begin{bmatrix} x \\ \frac{dx}{dt} \\ \theta \\ \frac{d\theta}{dt} \end{bmatrix} \quad (14)$$

which means that

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} \frac{dx}{dt} \\ \frac{d^2x}{dt^2} \\ \frac{d\theta}{dt} \\ \frac{d^2\theta}{dt^2} \end{bmatrix} \quad (15)$$

As mentioned in the problem statement, $u = F$. To find $\vec{f}(\vec{x}, u)$, we can use the physics equations in eq. (12) and eq. (13) to write

$$\vec{f}(\vec{x}, u) = \begin{bmatrix} x_2 \\ \frac{1}{1+\sin^2(x_3)}(u + \sin(x_3)(x_4^2 - g \cos(x_3))) \\ x_4 \\ \frac{1}{1+\sin^2(x_3)}(-u \cos(x_3) - x_4^2 \cos(x_3) \sin(x_3) + 2g \sin(x_3)) \end{bmatrix} \quad (16)$$

- (b) What point \vec{x}_* and u_* do we want to linearize around?

(HINT: Think about what we want the cart-pole system to do. What state do we want the system to converge to?)

Solution: We want the cart to be located at the origin and we want the pole to be vertical (with neither the cart nor the pole moving). Hence, we want $x = 0$, $\frac{dx}{dt} = 0$, $\theta = 0$, and $\frac{d\theta}{dt} = 0$. Thus, we can choose \vec{x}_* to be

$$\vec{x}_* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (17)$$

If we choose \vec{x}_* as above, then $f_2(\vec{x}_*, u) = u$, so we need $u_* = 0$ (since we require $f_2(\vec{x}_*, u_*) = 0$).

- (c) Write $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$. Find the Jacobian of $\vec{f}(\vec{x}, u)$. That is, find $J_{\vec{x}}\vec{f}$ and $J_u\vec{f}$. You may leave

your answer in terms of the components of \vec{x} , $\frac{\partial f_2}{\partial x_3}$, and $\frac{\partial f_4}{\partial x_3}$.

Solution: We know that

$$J_{\vec{x}} \vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} \end{bmatrix} \quad (18)$$

To compute so we have

$$J_{\vec{x}} \vec{f} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\partial f_2}{\partial x_3} & 2x_4 \frac{\sin(x_3)}{\sin^2(x_3)+1} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{\partial f_4}{\partial x_3} & -\frac{2x_4 \sin(x_3) \cos(x_3)}{\sin^2(x_3)+1} \end{bmatrix} \quad (19)$$

where $\frac{\partial f_2}{\partial x_3}$ and $\frac{\partial f_4}{\partial x_3}$ are written in full form below. Next, recall that

$$J_u \vec{f} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \\ \frac{\partial f_4}{\partial u} \end{bmatrix} \quad (20)$$

so we have

$$J_u \vec{f} = \begin{bmatrix} 0 \\ \frac{1}{1+\sin^2(x_3)} \\ 0 \\ -\frac{\cos(x_3)}{1+\sin^2(x_3)} \end{bmatrix} \quad (21)$$

(Optional) Computing $\frac{\partial f_2}{\partial x_3}$ and $\frac{\partial f_4}{\partial x_3}$:

The partial derivatives are as follows:

$$\frac{\partial f_2}{\partial x_3} = \frac{1}{(\sin^2(x_3) + 1)^2} \left(\cos(x_3) \left(x_4^2 (1 - \sin^2(x_3)) - 2u \sin(x_3) \right) \right) \quad (22)$$

$$+ g \sin^2(x_3) (\sin^2(x_3) + 1) + g \cos^2(x_3) (\sin^2(x_3) - 1) \quad (23)$$

$$\frac{\partial f_4}{\partial x_3} = \frac{1}{(\sin^2(x_3) + 1)^2} \left(\sin(x_3) (\sin^2(x_3) + 1) \left(x_4^2 \sin(x_3) + u \right) \right) \quad (24)$$

$$+ \cos^2(x_3) \left(x_4^2 \sin^2(x_3) - x_4^2 + 2u \sin(x_3) \right) - 2g (\sin^2(x_3) - 1) \cos(x_3) \quad (25)$$

Note: In this class, you will not be asked to compute derivatives as complicated as this.

- (d) **Linearize the dynamics about the \vec{x}_* that you found earlier. Explicitly write this linearized system.** You may use the fact that $\frac{\partial f_2}{\partial x_3}(\vec{x}_*, u_*) = -g$ and $\frac{\partial f_4}{\partial x_3}(\vec{x}_*, u_*) = 2g$. **Is the linearized system stable? How can we accomplish the task mentioned at the beginning of the problem?**

Solution: Plugging in $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$ into the expressions for $J_{\vec{x}} \vec{f}$ and $J_u \vec{f}$ from eq. (19) and eq. (21) respectively, we obtain

$$J_{\vec{x}} \vec{f}(\vec{x}_*, u_*) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2g & 0 \end{bmatrix} \quad (26)$$

$$J_u \vec{f}(\vec{x}_*, u_*) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad (27)$$

Define $A := J_{\vec{x}} \vec{f}(\vec{x}_*, u_*)$ and $B := J_u \vec{f}(\vec{x}_*, u_*)$. The linearized system is

$$\frac{d}{dt} \vec{x}(t) = A \vec{x}(t) + B \vec{u}(t) \quad (28)$$

The eigenvalues of A are 0 (repeated twice), $-\sqrt{2g}$ and $\sqrt{2g}$. The linearized system is not stable because of the eigenvalue $\sqrt{2g}$. To accomplish the given task, we may want to apply feedback control. It happens to be the case that, for this continuous time system, we can place the eigenvalues anywhere so long as they are distinct.

The accompanying [Colab notebook](#) shows a demo of how we may want to accomplish the goal of stabilizing the cart-pole system.

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