Discussion 7B

1. System Identification by Means of Least Squares

(a) Consider the scalar discrete-time system

$$x[i+1] = ax[i] + bu[i] + w[i] \tag{1}$$

Where the scalar state at timestep i is $x[i]$, the input applied at timestep i is $u[i]$ and $w[i]$ represents some (small) external disturbance that also participated at timestep i (which we cannot predict or control, it's a purely random disturbance).

Assume that you have measurements for the states $x[i]$ from $i = 0$ to ℓ and also measurements for the controls $u[i]$ from $i = 0$ to $\ell - 1$. Further assume $\ell \geq 2$.

Show that we can set up a linear system as in eq. (2) to find constants a and b. How do we solve this system?

$$\underbrace{\begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[\ell] \end{bmatrix}}_{\vec{s}} \approx \underbrace{\begin{bmatrix} x[0] & u[0] \\ x[1] & u[1] \\ \vdots & \vdots \\ x[\ell-1] & u[\ell-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{p}} \tag{2}$$

Solution: Our model is of the form

$$x[i+1] = ax[i] + bu[i] + w[i] \tag{3}$$

where $w[i]$ is our error term and we are interested in a and b. Since we cannot predict the disturbance $w[i]$ (and therefore cannot have a parameter in our solution associated with the effect of the disturbance on our system), we will solve the adjusted equation in eq. (4).

$$x[i+1] \approx ax[i] + bu[i] \tag{4}$$

We have measurements from $i = 1$ to $i = m$, and so our least squares formulation is:

$$\underbrace{\begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[\ell] \end{bmatrix}}_{\vec{s}} \approx \underbrace{\begin{bmatrix} x[0] & u[0] \\ x[1] & u[1] \\ \vdots & \vdots \\ x[\ell-1] & u[\ell-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{p}} \tag{5}$$

D is not necessarily a square matrix (it is tall), so we cannot invert it and solve for $\vec{p}$. Hence, we use least squares like previously mentioned. Thus, our best approximation for $\vec{p}$ is

$$\widehat{\vec{p}} = \left(D^\top D\right)^{-1} D^\top \vec{s} \tag{6}$$

Since we are using least squares, we can also group our estimation error (remember, $\widehat{\vec{p}} \neq \vec{p}$ necessarily) into $w[i]$.

(b) What if there were now two distinct scalar inputs to a scalar system

$$x[i+1] = ax[i] + b_1 u_1[i] + b_2 u_2[i] + w[i] \tag{7}$$

and that we have measurements as before, but now also for both of the control inputs.

Set up a least-squares problem that you can solve to get an estimate of the unknown system parameters a, b_1, b_2.

Solution: Our new model is of the form

$$x[i+1] = ax[i] + b_1 u_1[i] + b_2 u_2[i] + w[i] \tag{8}$$

where $w[i]$ is our error term and we are interested in a, b_1, b_2. As we did before, we will modify the system and drop the disturbance term, converting the equality to an approximation.

$$x[i+1] \approx ax[i] + b_1 u_1[i] + b_2 u_2[i] \tag{9}$$

As before, we have $[1, m]$ measurements, and so our least squares formulation is:

$$\underbrace{\begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[\ell] \end{bmatrix}}_{\vec{s}} \approx \underbrace{\begin{bmatrix} x[0] & u_1[0] & u_2[0] \\ x[1] & u_1[1] & u_2[1] \\ \vdots & \vdots & \vdots \\ x[\ell-1] & u_1[\ell-1] & u_2[\ell-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a \\ b_1 \\ b_2 \end{bmatrix}}_{\vec{p}} \tag{10}$$

(c) **What could go wrong in the previous case? For what kind of inputs would make least-squares fail to give you the parameters you want?**

Solution: We can take a look at the least squares formula, and think about what the possible failure points are.

$$\widehat{\vec{p}} = \left(D^\top D\right)^{-1} D^\top \vec{s}. \tag{11}$$

In this equation, the likely point of failure is the inversion of $D^\top D$; the other operations (matrix-matrix multiplications, matrix-vector multiplications) do not have the same issue.

$D^\top D$ might not be invertible when D has columns that are not linearly independent. For example, it could be because the inputs $\vec{u}_1$ and $\vec{u}_2$ are too similar, as if $\vec{u}_1 = \alpha \vec{u}_2$. We need these two inputs to be different and sufficiently varied so that least-squares does not fail.

(d) Now consider the two dimensional state case with a single input.

$$\vec{x}[i+1] = \begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \vec{x}[i] + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u[i] + \vec{w}[i] \tag{12}$$

How can we treat this like two parallel problems to set this up using least-squares to get estimates for the unknown parameters $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$? Write the least squares solution in terms of your known matrices and vectors (including based on the labels you gave to various matrices/vectors in previous parts). *Hint: What work/computation can we reuse across the two problems?*

Solution: We can rewrite eq. (12) as

$$\begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix} = \begin{bmatrix} a_{11}x_1[i] + a_{12}x_2[i] + b_1 u[i] \\ a_{21}x_1[i] + a_{22}x_2[i] + b_2 u[i] \end{bmatrix} \tag{13}$$

We can set up a problem to solve for a_{11}, a_{12}, b_1 (call this subsystem 1) and another problem to solve for a_{21}, a_{22}, b_2 (call this subsystem 2). We can rewrite the first row of eq. (13) as

$$x_1[i+1] = \begin{bmatrix} x_1[i] & x_2[i] & u[i] \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ b_1 \end{bmatrix} \tag{14}$$

and likewise for the second row

$$x_2[i+1] = \begin{bmatrix} x_1[i] & x_2[i] & u[i] \end{bmatrix} \begin{bmatrix} a_{21} \\ a_{22} \\ b_2 \end{bmatrix} \tag{15}$$

To find the unknowns in subsystem 1, we can set up the following least squares problem:

$$\underbrace{\begin{bmatrix} x_1[1] \\ x_1[2] \\ \vdots \\ x_1[\ell] \end{bmatrix}}_{\vec{s}_1} \approx \underbrace{\begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ x_1[1] & x_2[1] & u[1 \\ \vdots & \vdots & \vdots \\ x_1[\ell-1] & x_2[\ell-1] & u[\ell-1] \end{bmatrix}}_{D_1} \underbrace{\begin{bmatrix} a_{11} \\ a_{12} \\ b_1 \end{bmatrix}}_{\vec{p}_1} \tag{16}$$

Now, to find the unknowns in subsystem 2, we can set up the following least squares problem:

$$\underbrace{\begin{bmatrix} x_2[1] \\ x_2[2] \\ \vdots \\ x_2[\ell] \end{bmatrix}}_{\vec{s}_2} \approx \underbrace{\begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ x_1[1] & x_2[1] & u[1 \\ \vdots & \vdots & \vdots \\ x_1[\ell-1] & x_2[\ell-1] & u[\ell-1] \end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} a_{21} \\ a_{22} \\ b_2 \end{bmatrix}}_{\vec{p}_2} \tag{17}$$

Notice that $D_1 = D_2$. Hence, we can write $D = D_1 = D_2$, and we only need to compute $\left(D^\top D\right)^{-1} D^\top$ once. Hence, the solution for the ith subsystem (for $i \in \{1, 2\}$) is

$$\widehat{\vec{p}}_i = \left(D^\top D\right)^{-1} D^\top \vec{s}_i \tag{18}$$

Furthermore, we can horizontally stack the two separate problems for each subsystem as follows:

$$\underbrace{\begin{bmatrix} x_1[1] & x_2[1] \\ x_1[2] & x_2[2] \\ \vdots & \vdots \\ x_1[\ell] & x_2[\ell] \end{bmatrix}}_{S} \approx \underbrace{\begin{bmatrix} x_1[0] & x_2[0] & u[0] \\ x_1[1] & x_2[1] & u[1] \\ \vdots & \vdots & \vdots \\ x_1[\ell-1] & x_2[\ell-1] & u[\ell-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ b_1 & b_2 \end{bmatrix}}_{P} \tag{19}$$

Finally, solving this as a single least squares problem gives us

$$\widehat{P} = \left(D^\top D\right)^{-1} D^\top S \tag{20}$$

Contributors:

- Anish Muthali.
- Neelesh Ramachandran.
- Anant Sahai.
- Regina Eckert.
- Kareem Ahmad.