

1. Changing Coordinates and Systems of Differential Equations

A matrix differential equation (without input) follows the form

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \tag{1}$$

where $\vec{x}(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. Note that $\frac{d}{dt}\vec{x}(t)$ is equivalent to $\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \vdots \\ \frac{dx_n(t)}{dt} \end{bmatrix}$ where $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$.

In other words, taking the derivative of the vector is the same as taking the derivative elementwise of its components. When A is diagonal, we can treat the matrix differential equation as a system of n separate, scalar differential equations. In this discussion, we will use **change of variables** to tackle a matrix differential equation where A is not diagonal. This will help us model the behavior of more complex circuits where A will usually be non-diagonal.

First, we can practice by solving a matrix differential equation with diagonal A . Suppose we have the following differential equation (valid for $t \geq 0$)

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix} \vec{x}(t) \tag{2}$$

with initial condition $\vec{x}(0) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

- (a) **Write the matrix differential equation as a system of individual, scalar differential equations and solve for $\vec{x}(t)$ for $t \geq 0$.**

Now, suppose we had the following circuit with two capacitors. Let's write a system of differential equations involving the voltages across the capacitors (V_{C_1} and V_{C_2}).

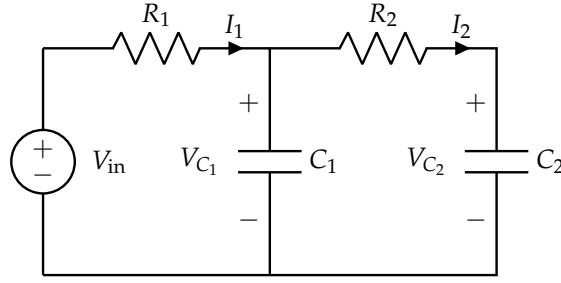


Figure 1: Two dimensional system: a circuit with two capacitors.

We start by solving for the currents and voltages across the capacitors:

$$V_{C_2} = V_{C_1} - I_2 R_2, \quad I_2 = C_2 \frac{d}{dt} V_{C_2} \quad (3)$$

$$V_{in} - I_1 R_1 = V_{C_1}, \quad I_1 = I_2 + C_1 \frac{d}{dt} V_{C_1} \quad (4)$$

This yields

$$I_1 = \frac{V_{in}}{R_1} - \frac{V_{C_1}}{R_1}, \quad I_2 = \frac{V_{C_1}}{R_2} - \frac{V_{C_2}}{R_2} \quad (5)$$

Now, we can plug into the formula for current across a capacitor:

$$\frac{dV_{C_1}}{dt} = \frac{1}{C_1} (I_1 - I_2) \quad (6)$$

$$= \frac{1}{C_1} \left(\frac{V_{in}}{R_1} - \frac{V_{C_1}}{R_1} - \frac{V_{C_1}}{R_2} + \frac{V_{C_2}}{R_2} \right) \quad (7)$$

$$= - \left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1} \right) V_{C_1} + \frac{V_{C_2}}{R_2 C_1} + \frac{V_{in}}{R_1 C_1} \quad (8)$$

$$\frac{dV_{C_2}}{dt} = \frac{1}{C_2} (I_2) \quad (9)$$

$$= \frac{V_{C_1}}{R_2 C_2} - \frac{V_{C_2}}{R_2 C_2} \quad (10)$$

Suppose we have the following values: $C_1 = 1 \mu\text{F}$, $C_2 = \frac{1}{3} \mu\text{F}$, $R_1 = \frac{1}{3} \text{M}\Omega$, $R_2 = \frac{1}{2} \text{M}\Omega$, $V_{in} = 0\text{V}$.

If we use these values for the resistors, capacitors, and input, we get:

$$\frac{dV_{C_1}(t)}{dt} = -5V_{C_1}(t) + 2V_{C_2}(t) \quad (11)$$

$$\frac{dV_{C_2}(t)}{dt} = 6V_{C_1}(t) - 6V_{C_2}(t) \quad (12)$$

Notice that we have a system of coupled differential equations (differential equation that depend on each other)! One way to solve this would be to relate V_{C_1} and V_{C_2} to each other to create a 2nd order differential equation. However, in this discussion, we will look to approach this problem using vector differential equations.

For the rest of this problem, let $x_1(t) = V_{C_1}(t)$ and $x_2(t) = V_{C_2}(t)$. Rewriting our equations in terms of x_1 and x_2 , we have:

$$\frac{dx_1(t)}{dt} = -5x_1(t) + 2x_2(t) \quad (13)$$

$$\frac{dx_2(t)}{dt} = 6x_1(t) - 6x_2(t) \quad (14)$$

- (b) **Write out the above system of differential equations in matrix form.** *HINT: Define the matrix differential equation in terms of $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. What is your “A” matrix here? Can we solve this system in a similar way as we did above?*

As you may have noticed, it is not possible to solve the differential equation using methods we have already covered in this class. We can try to use change of variables to turn this problem into one with a diagonal system since we know how to solve these types of equations. Consider the strategy outlined in fig. 2. We want to change variables to $\vec{z}(t)$, such that we end up with a differential equation where \tilde{A} will be diagonal. This is especially important when there is no clear path to a solution with just the system involving $\vec{x}(t)$ (as is the case here).

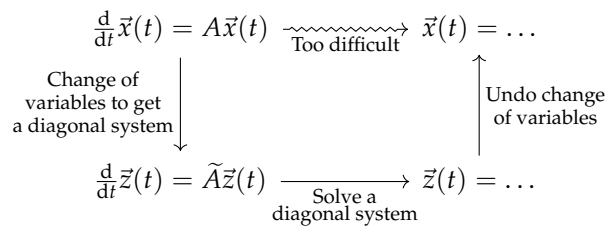


Figure 2: A Strategy to Solve for $\vec{x}(t)$

- (c) We can define $\vec{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$ to achieve the goal described above. Consider the following relationship between $\vec{x}(t)$ and $\vec{z}(t)$:

$$x_1(t) = -z_1(t) + 2z_2(t) \quad (15)$$

$$x_2(t) = 2z_1(t) + 3z_2(t) \quad (16)$$

Write out this transformation in matrix form ($\vec{x}(t) = V\vec{z}(t)$ for some V). This will give us a representation for $\vec{x}(t)$ in terms of $\vec{z}(t)$. **Then, find a way to represent $\vec{z}(t)$ in terms of $\vec{x}(t)$.** What conditions need to hold for this representation to work? Recall that, the inverse of a 2×2 matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (17)$$

As you have learned, in order to decouple the equations, V must be chosen such that its columns are the eigenvectors of our system matrix (A). You may want to practice finding V on your own. We are providing it for you here in the interest of time.

- (d) Suppose that the following initial conditions are given: $\vec{x}(0) = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$. Now that we changed variables, we also have to appropriately change our initial condition for the new variables we defined. **How do these initial conditions for $\vec{x}(t)$ translate into the initial conditions for $\vec{z}(t)$?** *HINT: Use the representation for $\vec{z}(t)$ in terms of $\vec{x}(t)$ from part 1.c.*

- (e) Now, we are ready to write a new differential equation for $\vec{z}(t)$. **Incorporate $\vec{z}(t)$, your new variable, into the matrix differential equation from part 1.b to come up with a differential equation for $\vec{z}(t)$.** *HINT: How can we substitute the $\vec{x}(t)$ terms with terms involving $\vec{z}(t)$? Also, recall that, since the derivative operator is linear, we can write $\frac{d}{dt} M\vec{x}(t) = M \frac{d}{dt} \vec{x}(t)$ where M is a matrix of constants. Can we solve this system of differential equations?*

- (f) **Solve the differential equation for $\vec{z}(t)$. Then, “undo” the change of variables from the previous parts to find a solution for $\vec{x}(t)$.** *HINT: How can we “recover” $\vec{x}(t)$ from $\vec{z}(t)$? Then, fill in the strategy diagram in fig. 3 with the following:*
- (i) Mathematically, how did we define our change of variables?
 - (ii) In terms of $\vec{z}(t)$, A , and V , what matrix differential equation did we solve?
 - (iii) Mathematically, how did we “undo” our change of variables?

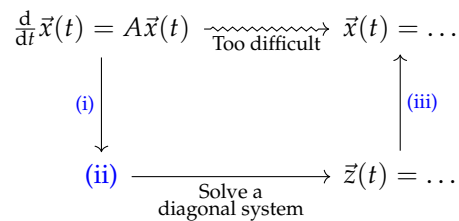


Figure 3: Mathematical Description of Our Strategy to Solve for $\vec{x}(t)$

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