Discussion 7A

1. Translating System of Differential Equations from Continuous Time to Discrete Time

Oftentimes, we wish to apply controls model on a computer. However, modeling a continuous time system on a computer is a nontrivial problem. Hence, we turn to discretizing our controls problem. That is, we define a discretized state $\vec{x}_d[i]$ and a discretized input $\vec{u}_d[i]$ that we "sample" every Δ seconds.

(a) Consider the scalar system below:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + bu(t). \tag{1}$$

where $x(t)$ is our state and $u(t)$ is our control input. Let $\lambda \neq 0$ be an arbitrary constant. Further suppose that our input $u(t)$ is piecewise constant, and that $x(t)$ is differentiable everywhere (and thus, continuous everywhere). That is, we define an interval $t \in [i\Delta, (i+1)\Delta)$ such that $u(t)$ is constant over this interval. Mathematically, we write this as

$$u(t) = u(i\Delta) = u_d[i] \text{ if } t \in [i\Delta, (i+1)\Delta). \tag{2}$$

The now-discretized input $u_d[i]$ is the same as the original input where we only "observe" a change in $u(t)$ every Δ seconds. Similarly, for $x(t)$,

$$x(t) = x(i\Delta) = x_d[i] \tag{3}$$

Let's revisit the solution for eq. (1), when we're given the initial conditions at t_0, i.e we know the value of $x(t_0)$ and want to solve for $x(t)$ at any time $t \geq t_0$:

$$x(t) = \mathrm{e}^{\lambda(t-t_0)}x(t_0) + b\int_{t_0}^{t} u(\theta)\mathrm{e}^{\lambda(t-\theta)}\,\mathrm{d}\theta \tag{4}$$

Given that we start at $t = i\Delta$, where $x(t) = x_d[i]$ is known, and satisfy eq. (1), where do we end up at $x_d[i+1]$? *(HINT): Think about the initial condition here. Where does our solution "start"?*

Solution: For $t \in [i\Delta, (i+1)\Delta)$, the differential equation takes the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + bu(t) = \lambda x(t) + bu_d[i] \tag{5}$$

where we choose our initial condition to be $x(i\Delta) = x_d[i]$, since this is a known quantity. We can solve this equation for $x(t)$ using the integral equation from eq. (4) and the fact that $u_d[i]$ is a constant value over this interval. In particular, we get the following form

$$x(t) = \mathrm{e}^{\lambda(t-i\Delta)}\underbrace{x(i\Delta)}_{x_d[i]} + b\int_{i\Delta}^{t}\underbrace{u(i\Delta)}_{u_d[i]}\mathrm{e}^{\lambda(t-\theta)}\,\mathrm{d}\theta \tag{6}$$

$$= \mathrm{e}^{\lambda(t-i\Delta)}x_d[i] + bu_d[i]\int_{i\Delta}^{t}\mathrm{e}^{\lambda(t-\theta)}\,\mathrm{d}\theta \tag{7}$$

Plugging in the timestep of interest, we set $t = (i+1)\Delta$, to evaluate $x_d[i+1]$ as

$$x_d[i+1] = x((i+1)\Delta) \tag{8}$$

$$= e^{\lambda\Delta} x_d[i] + b u_d[i] \int_{i\Delta}^{(i+1)\Delta} e^{\lambda((i+1)\Delta-\theta)}\, d\theta \tag{9}$$

$$= e^{\lambda\Delta} x_d[i] + b u_d[i] \frac{e^{\lambda\Delta} - e^0}{\lambda} \tag{10}$$

$$= e^{\lambda\Delta} x_d[i] + b u_d[i] \frac{e^{\lambda\Delta} - 1}{\lambda} \tag{11}$$

which gives us the solution for $x_d[i+1]$.

(b) Suppose we now have a continuous-time system of differential equations, that forms a vector differential equation. We express this with an input in vector form:

$$\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) + \vec{b}u(t) \tag{12}$$

where $\vec{x}(t)$ is n-dimensional. Suppose further that the matrix A has distinct and non-zero eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. with corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. We collect the eigenvectors together and form the matrix $V = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n]$.

We now wish to find a matrix A_d and a vector $\vec{b}_d$ such that

$$\vec{x}_d[i+1] = A_d \vec{x}_d[i] + \vec{b}_d u_d[i] \tag{13}$$

where $\vec{x}_d[i] = \vec{x}(i\Delta)$.

Firstly, define terms

$$e^{\Lambda\Delta} = \begin{bmatrix} e^{\lambda_1\Delta} & 0 & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & e^{\lambda_n\Delta} \end{bmatrix} \tag{14}$$

$$\Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & \frac{1}{\lambda_n} \end{bmatrix} \tag{15}$$

$$\vec{\tilde{u}}_d[i] = V^{-1}\vec{b}u_d[i] \tag{16}$$

Note that the term $e^{\Lambda\Delta}$ is just a label for our intents and purposes — this is not the same as applying e^x to every element in the matrix Λ.

Complete the following steps to derive a discretized system:

i. **Diagonalize the continuous time system using a change of variables (change of basis) to achieve a new system for $\vec{y}(t)$.**

ii. **Solve the diagonalized system. Remember that we only want a solution over the interval $t \in [i\Delta, (i+1)\Delta)$. Use the value at $t = i\Delta$ as your initial condition.**

iii. **Discretize the diagonalized system to obtain $\vec{y}_d[i]$. Show that**

$$\vec{y}_d[i+1] = \underbrace{\begin{bmatrix} e^{\lambda_1\Delta} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & e^{\lambda_n\Delta} \end{bmatrix}}_{e^{\Lambda\Delta}} \vec{y}_d[i] + \begin{bmatrix} \frac{e^{\lambda_1\Delta}-1}{\lambda_1} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \frac{e^{\lambda_n\Delta}-1}{\lambda_n} \end{bmatrix} \vec{\tilde{u}}_d[i] \tag{17}$$

Then, show that the matrix $\begin{bmatrix} \frac{e^{\lambda_1\Delta}-1}{\lambda_1} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \frac{e^{\lambda_n\Delta}-1}{\lambda_n} \end{bmatrix}$ **can be compactly written as $\Lambda^{-1}(e^{\Lambda\Delta} - I)$.**

iv. **Undo the change of variables on the discretized diagonal system to get the discretized solution of the original system.**

Solution:

i. First, following the hint, we notice that with a full set of distinct eigenvalues and corresponding eigenvectors, we can change coordinates so that $\vec{x}(t) = V\vec{y}(t)$ and $\vec{y}(t) = V^{-1}\vec{x}(t)$. Using this transformation we diagonalize the system of differential equations, i.e

$$\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) + \vec{b}u(t) \tag{18}$$

$$\implies \frac{dV\vec{y}(t)}{dt} = AV\vec{y}(t) + \vec{b}u(t) \tag{19}$$

$$\therefore \frac{d\vec{y}(t)}{dt} = V^{-1}AV\vec{y}(t) + V^{-1}\vec{b}u(t) \tag{20}$$

Note that using the basis of eigenvectors V, we've diagonalized A to get $\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \lambda_n \end{bmatrix}$

$$\therefore \frac{d\vec{y}(t)}{dt} = \Lambda\vec{y}(t) + V^{-1}\vec{b}u(t) \tag{21}$$

ii. Now, we can use the fact that we care about the solution for $\vec{y}(t)$ over the interval $t \in (i\Delta, (i+1)\Delta]$, so $u(t)$ is a constant. Thus, we can write eq. (21) as follows:

$$\frac{d\vec{y}(t)}{dt} = \Lambda\vec{y}(t) + \underbrace{V^{-1}\vec{b}u_d[i]}_{\vec{\tilde{u}}_d[i]} \tag{22}$$

Notice that this system is diagonal (and hence we can write it as a system of n differential equations). We can look at the kth differential equation. We will use the subscripting notation $(\vec{y}(t))_k$ and $\left(\vec{\tilde{u}}_d[i]\right)_k$ to denote the kth element of $\vec{y}(t)$ and $\vec{\tilde{u}}_d[i]$ respectively:

$$\frac{d(\vec{y}(t))_k}{dt} = \lambda_k(\vec{y}(t))_k + \left(\vec{\tilde{u}}_d[i]\right)_k \tag{23}$$

We can pattern match to the solution in eq. (7), setting $\lambda \to \lambda_k$, $u_d[i] \to \left(\vec{\tilde{u}}_d[i]\right)_k$, $b \to 1$, and $x(t) \to (\vec{y}(t))_k$, to get

$$(\vec{y}(t))_k = e^{\lambda_k(t-i\Delta)}(\vec{y}(i\Delta))_k + (\tilde{u}_d[i])_k \int_{i\Delta}^{t} e^{\lambda_k(t-\theta)}\, d\theta \tag{24}$$

for $t \in (i\Delta, (i+1)\Delta]$.

iii. Now, we want to find $(\vec{y}_d[i+1])_k = (\vec{y}((i+1)\Delta))_k$, so we can plug in for $t = (i+1)\Delta$ in eq. (24) and we will get

$$(\vec{y}_d[i+1])_k = (\vec{y}((i+1)\Delta))_k = e^{\lambda_k\Delta}(\vec{y}(i\Delta))_k + \left(\frac{e^{\lambda_k\Delta}-1}{\lambda_k}\right)\left(\vec{\tilde{u}}_d[i]\right)_k \tag{25}$$

Since we have a solution for the kth differential equation in the system, we can arrange all the differential equations in this system in matrix form as follows:

$$\underbrace{\vec{y}((i+1)\Delta)}_{\vec{y}_d[i+1]} = \underbrace{\begin{bmatrix} e^{\lambda_1\Delta} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & e^{\lambda_n\Delta} \end{bmatrix}}_{e^{\Lambda\Delta}} \underbrace{\vec{y}(i\Delta)}_{\vec{y}_d[i]} + \begin{bmatrix} \frac{e^{\lambda_1\Delta}-1}{\lambda_1} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \frac{e^{\lambda_n\Delta}-1}{\lambda_n} \end{bmatrix} \vec{\tilde{u}}_d[i] \tag{26}$$

Using the notation in the hint, we can write the second matrix in eq. (26) as:[1]

$$\begin{bmatrix} \frac{e^{\lambda_1\Delta}-1}{\lambda_1} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \frac{e^{\lambda_n\Delta}-1}{\lambda_n} \end{bmatrix} = \begin{bmatrix} \frac{e^{\lambda_1\Delta}}{\lambda_1} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \frac{e^{\lambda_n\Delta}}{\lambda_n} \end{bmatrix} + \begin{bmatrix} \frac{-1}{\lambda_1} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \frac{-1}{\lambda_n} \end{bmatrix} \tag{27}$$

$$= \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \frac{1}{\lambda_n} \end{bmatrix} \begin{bmatrix} e^{\lambda_1\Delta} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & e^{\lambda_n\Delta} \end{bmatrix} - \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \frac{1}{\lambda_n} \end{bmatrix} \tag{28}$$

$$= \Lambda^{-1}e^{\Lambda\Delta} - \Lambda^{-1}I \tag{29}$$

$$= \Lambda^{-1}\left(e^{\Lambda\Delta} - I\right) \tag{30}$$

This gives us

$$\vec{y}_d[i+1] = \vec{y}((i+1)\Delta) = e^{\Lambda\Delta}\underbrace{\vec{y}(i\Delta)}_{\vec{y}_d[i]} + \Lambda^{-1}\left(e^{\Lambda\Delta} - I\right)\vec{\tilde{u}}_d[i] \tag{31}$$

[1] In a matrix product, if both matrices are diagonal, the product is commutative.

iv. Recall that $\vec{x}(t) = V\vec{y}(t)$ so $\vec{x}_d[i] = \vec{x}(i\Delta) = V\vec{y}(i\Delta) = V\vec{y}_d[i]$, and likewise, $\vec{y}_d[i] = V^{-1}\vec{x}_d[i]$. Using this form in the simplification, we find that:

$$\begin{align}
\vec{x}_d[i+1] &= V\vec{y}_d[i+1] \tag{32}\\
&= V\left(e^{\Lambda\Delta}\vec{y}_d[i] + \Lambda^{-1}\left(e^{\Lambda\Delta} - I\right)\vec{\tilde{u}}_d[i]\right) \tag{33}\\
&= \left(Ve^{\Lambda\Delta}V^{-1}\right)\vec{x}_d[i] + \left(V\Lambda^{-1}\left(e^{\Lambda\Delta} - I\right)\right)\vec{\tilde{u}}_d[i] \tag{34}
\end{align}$$

Now, recall that our original goal was to write out A_d and $\vec{b}_d$, and we can do that now with our expression. Re-substituting $\vec{\tilde{u}}_d[i] = V^{-1}\vec{b}u_d[i]$ we have:

$$\begin{align}
\vec{x}_d[i+1] &= \left(Ve^{\Lambda\Delta}V^{-1}\right)\vec{x}_d[i] + \left(V\Lambda^{-1}\left(e^{\Lambda\Delta} - I\right)\right)V^{-1}\vec{b}u_d[i] \tag{35}\\
&= \underbrace{\left(Ve^{\Lambda\Delta}V^{-1}\right)}_{A_d}\vec{x}_d[i] + \underbrace{\left(V\Lambda^{-1}\left(e^{\Lambda\Delta} - I\right)V^{-1}\vec{b}\right)}_{\vec{b}_d}u_d[i] \tag{36}
\end{align}$$

(c) Consider the discrete-time system

$$\vec{x}_d[i+1] = A_d\vec{x}_d[i] + \vec{b}_d u_d[i] \tag{37}$$

Suppose that $\vec{x}_d[0] = \vec{x}_0$. **Unroll the implicit recursion and show that the solution follows the form in eq. (38).**

$$\vec{x}_d[i] = A_d^i \vec{x}_d[0] + \left(\sum_{j=0}^{i-1} u_d[j] A_d^{i-1-j}\right)\vec{b}_d \tag{38}$$

You may want to verify that this guess works by checking the form of $\vec{x}_d[i+1]$. You don't need to worry about what A_d and $\vec{b}_d$ actually are in terms of the original parameters.

(Hint: If we have a scalar difference equation, how would you solve the recurrence? Try writing $\vec{x}_d[i]$ in terms of $\vec{x}_d[0]$ for $i = 1, 2, 3$ and look for a pattern.)

Solution: Here, we derive the unrolled recursion and make a guess at the form of the solution in summation notation. Let's look at the pattern starting with $\vec{x}_d[1]$, given that $\vec{x}_d[i+1] = A_d\vec{x}_d[i] + \vec{b}_d u_d[i]$,

$$\begin{aligned}
\vec{x}_d[1] &= A_d\vec{x}_d[0] + \vec{b}_d u_d[0] && (39)\\
\vec{x}_d[2] &= A_d\vec{x}_d[1] + \vec{b}_d u_d[1] && (40)\\
&= A_d(A_d\vec{x}_d[0] + \vec{b}_d u_d[0]) + \vec{b}_d u_d[1] && (41)\\
&= A_d^2\vec{x}_d[0] + A_d\vec{b}_d u_d[0] + \vec{b}_d u_d[1] && (42)\\
\vec{x}_d[3] &= A_d\vec{x}_d[2] + \vec{b}_d u_d[2] && (43)\\
&= A_d\left(A_d^2\vec{x}_d[0] + A_d\vec{b}_d u_d[0] + \vec{b}_d u_d[1]\right) + \vec{b}_d u_d[2] && (44)\\
&= A_d^3\vec{x}_d[0] + A_d^2\vec{b}_d u_d[0] + A_d\vec{b}_d u_d[1] + \vec{b}_d u_d[2] && (45)
\end{aligned}$$

So, given this pattern, if we guess:

$$\vec{x}_d[i] = A_d^i \vec{x}_d[0] + \left(\sum_{j=0}^{i-1} u_d[j] A_d^{i-1-j}\right)\vec{b}_d \tag{46}$$

Then, let's see what we get for $\vec{x}_d[i+1]$, and make sure our guess is correct:

$$\begin{aligned}
\vec{x}_d[i+1] &= A_d\vec{x}_d[i] + \vec{b}_d u_d[i] && (47)\\
&= A_d\left(A_d^i\vec{x}_d[0] + \left(\sum_{j=0}^{i-1} u_d[j] A_d^{i-1-j}\right)\vec{b}_d\right) + \vec{b}_d u_d[i] && (48)\\
&= A_d^{i+1}\vec{x}_d[0] + \left(\left(\sum_{j=0}^{i-1} u_d[j] A_d^{i-j}\right) + u_d[i]\right)\vec{b}_d && (49)\\
&= A_d^{i+1}\vec{x}_d[0] + \left(\sum_{j=0}^{i} u_d[j] A_d^{i-j}\right)\vec{b}_d && (50)
\end{aligned}$$

This satisfies (46), for $i+1$ and hence our guess was correct!

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