1. Translating System of Differential Equations from Continuous Time to Discrete Time

Oftentimes, we wish to apply controls model on a computer. However, modeling a continuous time system on a computer is a nontrivial problem. Hence, we turn to discretizing our controls problem. That is, we define a discretized state $x_d[i]$ and a discretized input $u_d[i]$ that we “sample” every $\Delta$ seconds.

(a) Consider the scalar system below:

$$\frac{dx(t)}{dt} = \lambda x(t) + bu(t). \tag{1}$$

where $x(t)$ is our state and $u(t)$ is our control input. Let $\lambda \neq 0$ be an arbitrary constant. Further suppose that our input $u(t)$ is piecewise constant, and that $x(t)$ is differentiable everywhere (and thus, continuous everywhere). That is, we define an interval $t \in [i\Delta, (i+1)\Delta)$ such that $u(t)$ is constant over this interval. Mathematically, we write this as

$$u(t) = u(i\Delta) = u_d[i] \text{ if } t \in [i\Delta, (i+1)\Delta). \tag{2}$$

The now-discretized input $u_d[i]$ is the same as the original input where we only “observe” a change in $u(t)$ every $\Delta$ seconds. Similarly, for $x(t)$,

$$x(t) = x(i\Delta) = x_d[i] \tag{3}$$

Let’s revisit the solution for eq. (1), when we’re given the initial conditions at $t_0$, i.e we know the value of $x(t_0)$ and want to solve for $x(t)$ at any time $t \geq t_0$:

$$x(t) = e^{\lambda(t-t_0)}x(t_0) + b \int_{t_0}^{t} u(\theta)e^{\lambda(t-\theta)} \, d\theta \tag{4}$$

Given that we start at $t = i\Delta$, where $x(t) = x_d[i]$ is known, and satisfy eq. (1), where do we end up at $x_d[i + 1]$? (HINT): Think about the initial condition here. Where does our solution “start”?

**Solution:** For $t \in [i\Delta, (i+1)\Delta)$, the differential equation takes the form

$$\frac{dx(t)}{dt} = \lambda x(t) + bu(t) = \lambda x(t) + bu_d[i] \tag{5}$$

where we choose our initial condition to be $x(i\Delta) = x_d[i]$, since this is a known quantity. We can solve this equation for $x(t)$ using the integral equation from eq. (4) and the fact that $u_d[i]$ is a constant value over this interval. In particular, we get the following form

$$x(t) = e^{\lambda(t-i\Delta)}x(i\Delta) + b \int_{i\Delta}^{t} u(i\Delta) e^{\lambda(t-\theta)} \, d\theta \tag{6}$$

$$= e^{\lambda(t-i\Delta)}x_d[i] + bu_d[i] \int_{i\Delta}^{t} e^{\lambda(t-\theta)} \, d\theta \tag{7}$$

Plugging in the timestep of interest, we set $t = (i+1)\Delta$, to evaluate $x_d[i + 1]$ as

$$x_d[i + 1] = x((i+1)\Delta) \tag{8}$$
\[ e^{\Delta x_d[i]} + bu_d[i] \int_{\Delta i}^{(i+1)\Delta} e^{\lambda(i+1)\Delta - \theta} d\theta \]
\[ = e^{\lambda x_d[i]} + bu_d[i] e^{\lambda \Delta - e^0} \]
\[ = e^{\lambda x_d[i]} + bu_d[i] e^{\lambda \Delta - 1} \]

which gives us the solution for \( x_d[i + 1] \).

(b) Suppose we now have a continuous-time system of differential equations, that forms a vector differential equation. We express this with an input in vector form:

\[ \frac{d\vec{x}(t)}{dt} = A \vec{x}(t) + \vec{b}u(t) \]

where \( \vec{x}(t) \) is \( n \)-dimensional. Suppose further that the matrix \( A \) has distinct and non-zero eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) with corresponding eigenvectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \). We collect the eigenvectors together and form the matrix \( V = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n] \).

We now wish to find a matrix \( A_d \) and a vector \( \vec{b}_d \) such that

\[ \vec{x}_d[i + 1] = A_d \vec{x}_d[i] + \vec{b}_d u_d[i] \]

where \( \vec{x}_d[i] = \vec{x}(i\Delta) \).

Firstly, define terms

\[ e^{\Lambda \Delta} = \begin{bmatrix}
    e^{\lambda_1 \Delta} & 0 & \ldots & 0 \\
    \vdots & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & \ldots & \ldots & e^{\lambda_n \Delta}
\end{bmatrix} \]

\[ \Lambda^{-1} = \begin{bmatrix}
    \frac{1}{\lambda_1} & 0 & \ldots & 0 \\
    \vdots & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & \ldots & \ldots & \frac{1}{\lambda_n}
\end{bmatrix} \]

\[ \vec{\tilde{u}}_d[i] = V^{-1} \vec{b}u_d[i] \]

Note that the term \( e^{\Lambda \Delta} \) is just a label for our intents and purposes — this is not the same as applying \( e^{\lambda} \) to every element in the matrix \( \Lambda \).

Complete the following steps to derive a discretized system:

i. Diagonalize the continuous time system using a change of variables (change of basis) to achieve a new system for \( \tilde{y}(t) \).

ii. Solve the diagonalized system. Remember that we only want a solution over the interval \( t \in [i\Delta, (i + 1)\Delta) \). Use the value at \( t = i\Delta \) as your initial condition.
iii. Discretize the diagonalized system to obtain $\vec{y}_d[i]$. Show that

$$\vec{y}_d[i + 1] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \end{bmatrix} e^{\lambda_n \Delta} \vec{y}_d[i] + \begin{bmatrix} e^{\lambda_1 \Delta} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & e^{\lambda_n \Delta} & \cdots \end{bmatrix} \vec{u}_d[i]$$ (17)

Then, show that the matrix

$$\begin{bmatrix} e^{\lambda_1 \Delta - 1} \lambda_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & e^{\lambda_n \Delta - 1} \lambda_n & \cdots \end{bmatrix}$$

can be compactly written as $\Lambda^{-1} (e^{\Lambda \Delta} - I)$.

iv. Undo the change of variables on the discretized diagonal system to get the discretized solution of the original system.

Solution:

i. First, following the hint, we notice that with a full set of distinct eigenvalues and corresponding eigenvectors, we can change coordinates so that $\vec{x}(t) = V\vec{y}(t)$ and $\vec{y}(t) = V^{-1}\vec{x}(t)$.

Using this transformation we diagonalize the system of differential equations, i.e

$$\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) + \vec{b}u(t)$$ (18)

$$\Rightarrow \frac{dV\vec{y}(t)}{dt} = AV\vec{y}(t) + \vec{b}u(t)$$ (19)

$$\therefore \frac{d\vec{y}(t)}{dt} = V^{-1}AV\vec{y}(t) + V^{-1}\vec{b}u(t)$$ (20)

Note that using the basis of eigenvectors $V$, we’ve diagonalized $A$ to get $\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \lambda_n & \cdots \end{bmatrix}$

$$\therefore \frac{d\vec{y}(t)}{dt} = \Lambda\vec{y}(t) + V^{-1}\vec{b}u(t)$$ (21)

ii. Now, we can use the fact that we care about the solution for $\vec{y}(t)$ over the interval $t \in (i\Delta, (i + 1)\Delta]$, so $u(t)$ is a constant. Thus, we can write eq. (21) as follows:

$$\frac{d\vec{y}(t)}{dt} = \Lambda\vec{y}(t) + V^{-1}\vec{b}u_d[i]$$ (22)

Notice that this system is diagonal (and hence we can write it as a system of $n$ differential equations). We can look at the $k$th differential equation. We will use the subscripting notation $(\vec{y}(t))_k$ and $(\vec{u}_d[i])_k$ to denote the $k$th element of $\vec{y}(t)$ and $\vec{u}_d[i]$ respectively:

$$\frac{d((\vec{y}(t))_k)}{dt} = \lambda_k((\vec{y}(t))_k + (\vec{u}_d[i])_k$$ (23)
We can pattern match to the solution in eq. (7), setting \( \lambda \rightarrow \lambda_k \), \( u_d[i] \rightarrow \left( \vec{u}_d[i] \right) \), \( b \rightarrow 1 \), and \( x(t) \rightarrow (\vec{y}(t))_k \), to get

\[
(\vec{y}(t))_k = e^{\lambda_k (t-i\Delta)} (\vec{y}(i\Delta))_k + (\vec{u}_d[i])_k \int_{i\Delta}^t e^{\lambda_k (t-\theta)} \, d\theta
\]  

(24)

for \( t \in (i\Delta, (i+1)\Delta) \).

iii. Now, we want to find \( (\vec{y}_d[i+1])_k = (\vec{y}((i+1)\Delta))_k \) so we can plug in for \( t = (i+1)\Delta \) in eq. (24) and we will get

\[
(\vec{y}_d[i+1])_k = (\vec{y}((i+1)\Delta))_k = e^{\lambda_k \Delta} (\vec{y}(i\Delta))_k + \left( \frac{e^{\lambda_k \Delta} - 1}{\lambda_k} \right) (\vec{u}_d[i])_k
\]

(25)

Since we have a solution for the \( k \)th differential equation in the system, we can arrange all the differential equations in this system in matrix form as follows:

\[
\vec{y}((i+1)\Delta) = \begin{bmatrix} e^{\lambda_1 \Delta} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n \Delta} & \end{bmatrix} \begin{bmatrix} \vec{y}(i\Delta) \\ \vdots \\ \vdots \\ \vec{y}_d[i] \end{bmatrix} + \begin{bmatrix} e^{\lambda_1 \Delta} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n \Delta} & \end{bmatrix} (\vec{u}_d[i])_k
\]

(26)

Using the notation in the hint, we can write the second matrix in eq. (26) as:

\[
\begin{bmatrix} e^{\lambda_1 \Delta} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n \Delta} & \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\lambda_n} & \end{bmatrix} + \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & \end{bmatrix}
\]

(27)

\[
\begin{bmatrix} e^{\lambda_1 \Delta} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n \Delta} & \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & \end{bmatrix} - \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\lambda_n} & \end{bmatrix}
\]

(28)

\[
= \Lambda^{-1} e^{\Lambda \Delta} - \Lambda^{-1} I
\]

(29)

\[
= \Lambda^{-1} (e^{\Lambda \Delta} - I)
\]

(30)

This gives us

\[
\vec{y}_d[i+1] = \vec{y}((i+1)\Delta) = e^{\Lambda \Delta} (\vec{y}(i\Delta)) + \Lambda^{-1} (e^{\Lambda \Delta} - I) \vec{u}_d[i]
\]

(31)

\[1\text{In a matrix product, if both matrices are diagonal, the product is commutative.}\]
iv. Recall that $\vec{x}(t) = V \vec{y}(t)$ so $\vec{x}_d[i] = x(i\Delta) = V \vec{y}(i\Delta) = V \vec{y}_d[i]$, and likewise, $\vec{y}_d[i] = \vec{V}^{-1} \vec{y}_d[i]$. Using this form in the simplification, we find that:

$$\vec{x}_d[i+1] = V\vec{y}_d[i+1]$$

$$= V\left(e^{\Lambda \Delta}\vec{y}_d[i] + \Lambda^{-1}\left(e^{\Lambda \Delta} - I\right)\vec{u}_d[i]\right)$$

$$= \left(Ve^{\Lambda \Delta}V^{-1}\right)\vec{x}_d[i] + \left(V\Lambda^{-1}\left(e^{\Lambda \Delta} - I\right)V^{-1}\right)\vec{u}_d[i]$$

Now, recall that our original goal was to write out $A_d$ and $\vec{b}_d$, and we can do that now with our expression. Re-substituting $\vec{u}_d[i] = V^{-1}\vec{b}_d[i]$ we have:

$$\vec{x}_d[i+1] = \left(Ve^{\Lambda \Delta}V^{-1}\right)\vec{x}_d[i] + \left(V\Lambda^{-1}\left(e^{\Lambda \Delta} - I\right)V^{-1}\right)\vec{b}_d[i]$$

$$= \left(Ve^{\Lambda \Delta}V^{-1}\right)\vec{x}_d[i] + \left(V\Lambda^{-1}\left(e^{\Lambda \Delta} - I\right)V^{-1}\right)\vec{b}_d[i]$$
(c) Consider the discrete-time system

\[ \bar{x}_d[i + 1] = A_d \bar{x}_d[i] + \bar{b}_d u_d[i] \]  

(37)

Suppose that \( \bar{x}_d[0] = \bar{x}_0 \). Unroll the implicit recursion and show that the solution follows the form in eq. (38).

\[ \bar{x}_d[i] = A_d^i \bar{x}_d[0] + \left( \sum_{j=0}^{i-1} u_d[j] A_d^{i-1-j} \right) \bar{b}_d \]  

(38)

You may want to verify that this guess works by checking the form of \( \bar{x}_d[i + 1] \). You don’t need to worry about what \( A_d \) and \( \bar{b}_d \) actually are in terms of the original parameters.

(Hint: If we have a scalar difference equation, how would you solve the recurrence? Try writing \( \bar{x}_d[i] \) in terms of \( \bar{x}_d[0] \) for \( i = 1, 2, 3 \) and look for a pattern.)

**Solution:** Here, we derive the unrolled recursion and make a guess at the form of the solution in summation notation. Let’s look at the pattern starting with \( \bar{x}_d[1] \), given that \( \bar{x}_d[i + 1] = A_d \bar{x}_d[i] + \bar{b}_d u_d[i] \),

\[ \bar{x}_d[1] = A_d \bar{x}_d[0] + \bar{b}_d u_d[0] \]  

(39)

\[ \bar{x}_d[2] = A_d \bar{x}_d[1] + \bar{b}_d u_d[1] \]  

(40)

\[ = A_d (A_d \bar{x}_d[0] + \bar{b}_d u_d[0]) + \bar{b}_d u_d[1] \]  

(41)

\[ = A_d^2 \bar{x}_d[0] + A_d \bar{b}_d u_d[0] + \bar{b}_d u_d[1] \]  

(42)

\[ \bar{x}_d[3] = A_d \bar{x}_d[2] + \bar{b}_d u_d[2] \]  

(43)

\[ = A_d \left( A_d^2 \bar{x}_d[0] + A_d \bar{b}_d u_d[0] + \bar{b}_d u_d[1] \right) + \bar{b}_d u_d[2] \]  

(44)

\[ = A_d^3 \bar{x}_d[0] + A_d^2 \bar{b}_d u_d[0] + A_d \bar{b}_d u_d[1] + \bar{b}_d u_d[2] \]  

(45)

So, given this pattern, if we guess:

\[ \bar{x}_d[i] = A_d^i \bar{x}_d[0] + \left( \sum_{j=0}^{i-1} u_d[j] A_d^{i-1-j} \right) \bar{b}_d \]  

(46)

Then, let’s see what we get for \( \bar{x}_d[i + 1] \), and make sure our guess is correct:

\[ \bar{x}_d[i + 1] = A_d \bar{x}_d[i] + \bar{b}_d u_d[i] \]  

(47)

\[ = A_d \left( A_d^i \bar{x}_d[0] + \left( \sum_{j=0}^{i-1} u_d[j] A_d^{i-1-j} \right) \bar{b}_d \right) + \bar{b}_d u_d[i] \]  

(48)

\[ = A_d^{i+1} \bar{x}_d[0] + \left( \sum_{j=0}^{i-1} u_d[j] A_d^{i-j} \right) u_d[i] + \bar{b}_d u_d[i] \]  

(49)

\[ = A_d^{i+1} \bar{x}_d[0] + \left( \sum_{j=0}^{i} u_d[j] A_d^{i-j} \right) \bar{b}_d \]  

(50)

This satisfies (46), for \( i + 1 \) and hence our guess was correct!

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