Discussion 6C

The following note is useful for this discussion: Note 2j.

1. Complex Inner Products, Projections, and Orthonormality

In this discussion, we will show that the results we have already shown for projections are also applicable to complex vectors and the complex inner product, which will be discussed in lecture. This discussion is a preview of the lecture material on complex vectors and covers some fundamental properties. The complex inner product is defined as

$$\langle \vec{u}, \vec{v} \rangle = \vec{v}^* \vec{u} = \sum_{i=1}^{n} u_i \overline{v_i} \tag{1}$$

where $\vec{u}, \vec{v} \in \mathbb{C}^n$. From this inner product, we define the norm $\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$, which is always a real number.

(a) For complex vectors, we have the following projection formula:

$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle} \vec{u} \tag{2}$$

Confirm that $\text{proj}_{\vec{u}}(\alpha\vec{u}) = \alpha\vec{u}$**, where** $\alpha \in \mathbb{C}$**.**

Solution: We can apply the projection formula to obtain

$$\begin{aligned}
\text{proj}_{\vec{u}}(\alpha\vec{u}) &= \frac{\langle \alpha\vec{u}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle} \vec{u} && (3)\\
&= \frac{\vec{u}^*(\alpha\vec{u})}{\vec{u}^*\vec{u}} \vec{u} && (4)\\
&= \frac{\alpha(\vec{u}^*\vec{u})}{\vec{u}^*\vec{u}} \vec{u} && (5)\\
&= \alpha\vec{u} && (6)
\end{aligned}$$

(b) Define orthogonality the same way that we did with real vectors. That is, two vectors $\vec{u}$ and $\vec{v}$ are orthogonal if $\langle \vec{u}, \vec{v} \rangle = \vec{v}^* \vec{u} = 0$.

Show the orthogonality principle of projections, in that $\langle \vec{u}, \vec{v} - \text{proj}_{\vec{u}}(\vec{v}) \rangle = 0$**. Is it also true that** $\langle \vec{v} - \text{proj}_{\vec{u}}(\vec{v}), \vec{u} \rangle = 0$**?**

Solution: Using the definition of the projection and inner product, we have

$$\begin{aligned}
\langle \vec{u}, \vec{v} - \text{proj}_{\vec{u}}(\vec{v}) \rangle &= (\vec{v} - \text{proj}_{\vec{u}}(\vec{v}))^* \vec{u} && (7)\\
&= \left(\vec{v}^* - \left(\frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle} \right)^* \vec{u}^* \right) \vec{u} && (8)\\
&= \vec{v}^* \vec{u} - \left(\frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle} \right)^* \vec{u}^* \vec{u} && (9)
\end{aligned}$$

$$= \vec{v}^*\vec{u} - (\langle \vec{v},\, \vec{u}\rangle)^* \underbrace{\frac{\langle \vec{u},\, \vec{u}\rangle}{\langle \vec{u},\, \vec{u}\rangle}}_{1} \tag{10}$$

$$= \vec{v}^*\vec{u} - (\vec{u}^*\vec{v})^* \tag{11}$$

$$= \vec{v}^*\vec{u} - \vec{v}^*\vec{u} = 0 \tag{12}$$

where we use the fact that the norm is real to obtain eq. (10).

We know that $\langle \vec{x},\, \vec{y}\rangle = (\langle \vec{y},\, \vec{x}\rangle)^*$ for any arbitrary vectors $\vec{x}, \vec{y} \in \mathbb{C}^n$. Hence,

$$\langle \vec{v} - \text{proj}_{\vec{u}}(\vec{v}),\, \vec{u}\rangle = (\langle \vec{u},\, \vec{v} - \text{proj}_{\vec{u}}(\vec{v})\rangle)^* = (0)^* = 0 \tag{13}$$

Hence, we can either show $\langle \vec{u},\, \vec{v} - \text{proj}_{\vec{u}}(\vec{v})\rangle = 0$ or $\langle \vec{v} - \text{proj}_{\vec{u}}(\vec{v}),\, \vec{u}\rangle = 0$, and it would be an equally valid proof.

(c) We can try to generalize the idea of projections to least squares. Let's say we want to project onto the column space of a matrix $A \in \mathbb{C}^{m\times n}$, which has full column rank (so it must be that $m \geq n$). The formula for this is

$$\text{proj}_{\text{Col}(A)}(\vec{u}) = A(A^*A)^{-1}A^*\vec{u} \tag{14}$$

where $\vec{u} \in \mathbb{C}^m$.

Confirm that, if $\vec{u} = A\vec{x}$ for some $\vec{x} \in \mathbb{C}^n$, then $\text{proj}_{\text{Col}(A)}(\vec{u}) = \vec{u}$.

Solution: Substituting $\vec{u} = A\vec{x}$ into the least squares projection formula,

$$\text{proj}_{\text{Col}(A)}(\vec{u}) = A(A^*A)^{-1}A^*\vec{u} \tag{15}$$

$$= A(A^*A)^{-1}A^*(A\vec{x}) \tag{16}$$

$$= A\underbrace{(A^*A)^{-1}(A^*A)}_{I_{n\times n}}\vec{x} \tag{17}$$

$$= A\vec{x} = \vec{u} \tag{18}$$

(d) **Show the orthogonality principle for the least squares projection formula.**

Solution: Let $\vec{b} \in \text{Col}(A)$ be an arbitrary vector in A's column space. This means that there exists a vector $\vec{x} \in \mathbb{C}^n$ such that $A\vec{x} = \vec{b}$. We can show the orthogonality principle for the least squares projection formula by showing $\left\langle \vec{u} - \text{proj}_{\text{Col}(A)}(\vec{u}),\, \vec{b}\right\rangle = 0$. Hence,

$$\left\langle \vec{u} - \text{proj}_{\text{Col}(A)}(\vec{u}),\, \vec{b}\right\rangle = \vec{b}^*\left(\vec{u} - \text{proj}_{\text{Col}(A)}\left(\vec{b}\right)\right) \tag{19}$$

$$= (A\vec{x})^*\left(\vec{u} - A(A^*A)^{-1}A^*\vec{u}\right) \tag{20}$$

$$= \vec{x}^*A^*\vec{u} - \vec{x}^*\underbrace{A^*A(A^*A)^{-1}}_{I_{n\times n}}A^*\vec{u} \tag{21}$$

$$= \vec{x}^*A^*\vec{u} - \vec{x}^*A^*\vec{u} = 0 \tag{22}$$

(e) In the complex domain, we can define orthonormality similar to how we did with real vectors. That is, a collection of vectors $\{\vec{u}_1, \ldots, \vec{u}_m\}$ are orthonormal if

i. $\langle \vec{u}_i, \vec{u}_j \rangle = \vec{u}_j^* \vec{u}_i = 0$ for $i \neq j$ and

ii. $\|\vec{u}_i\| = \sqrt{\langle \vec{u}_i, \vec{u}_i \rangle} = 1$

for $i = 1, \ldots, n$.

Now let the columns of A be orthonormal. **Show $A^*A = I_{n\times n}$. Then, derive a simplified projection formula.**

(HINT: Consider writing $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$ where each $\vec{a}_i \in \mathbb{C}^m$ are orthonormal.)

Solution: Using the hint, we can write $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$. This means that $A^* = \begin{bmatrix} \vec{a}_1^* \\ \vdots \\ \vec{a}_n^* \end{bmatrix}$. Using this notation, the ijth entry of A^*A is

$$(A^*A)_{ij} = \vec{a}_i^* \vec{a}_j \tag{23}$$

This is 0 if $i \neq j$ and 1 if $i = j$, by the definition of orthonormality. Hence,

$$A^*A = I_{n\times n} \tag{24}$$

Plugging this into eq. (14), we have

$$\text{proj}_{\text{Col}(A)}(\vec{u}) = A(A^*A)^{-1}A^*\vec{u} = AA^*\vec{u} \tag{25}$$

Contributors:

- Anish Muthali.
- Druv Pai.
- Nathan Lambert.
- Anant Sahai.
- Kumar Krishna Agrawal.

