Discussion 6C

The following note is useful for this discussion: Note 2j.

1. Complex Inner Products, Projections, and Orthonormality

In this discussion, we will show that the results we have already shown for projections are also applicable to complex vectors and the complex inner product, which will be discussed in lecture. This discussion is a preview of the lecture material on complex vectors and covers some fundamental properties. The complex inner product is defined as

$$\langle \vec{u},\, \vec{v} \rangle = \vec{v}^* \vec{u} = \sum_{i=1}^{n} u_i \overline{v_i} \tag{1}$$

where $\vec{u}, \vec{v} \in \mathbb{C}^n$. From this inner product, we define the norm $\|\vec{u}\| = \sqrt{\langle \vec{u},\, \vec{u} \rangle}$, which is always a real number.

(a) For complex vectors, we have the following projection formula:

$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{\langle \vec{v},\, \vec{u} \rangle}{\langle \vec{u},\, \vec{u} \rangle} \vec{u} \tag{2}$$

Confirm that $\text{proj}_{\vec{u}}(\alpha \vec{u}) = \alpha \vec{u}$**, where** $\alpha \in \mathbb{C}$**.**

(b) Define orthogonality the same way that we did with real vectors. That is, two vectors $\vec{u}$ and $\vec{v}$ are orthogonal if $\langle \vec{u},\, \vec{v} \rangle = \vec{v}^* \vec{u} = 0$.

Show the orthogonality principle of projections, in that $\left\langle \vec{u},\, \vec{v} - \text{proj}_{\vec{u}}(\vec{v}) \right\rangle = 0$**. Is it also true that** $\left\langle \vec{v} - \text{proj}_{\vec{u}}(\vec{v}),\, \vec{u} \right\rangle = 0$**?**

(c) We can try to generalize the idea of projections to least squares. Let's say we want to project onto the column space of a matrix $A \in \mathbb{C}^{m\times n}$, which has full column rank (so it must be that $m \geq n$). The formula for this is

$$\text{proj}_{\text{Col}(A)}(\vec{u}) = A(A^*A)^{-1}A^*\vec{u} \tag{3}$$

where $\vec{u} \in \mathbb{C}^m$.

Confirm that, if $\vec{u} = A\vec{x}$ for some $\vec{x} \in \mathbb{C}^n$, then $\text{proj}_{\text{Col}(A)}(\vec{u}) = \vec{u}$.

(d) **Show the orthogonality principle for the least squares projection formula.**

(e) In the complex domain, we can define orthonormality similar to how we did with real vectors. That is, a collection of vectors $\{\vec{u}_1, \ldots, \vec{u}_m\}$ are orthonormal if

i. $\langle \vec{u}_i, \vec{u}_j \rangle = \vec{u}_j^* \vec{u}_i = 0$ for $i \neq j$ and

ii. $\|\vec{u}_i\| = \sqrt{\langle \vec{u}_i, \vec{u}_i \rangle} = 1$

for $i = 1, \ldots, n$.

Now let the columns of A be orthonormal. **Show $A^*A = I_{n\times n}$. Then, derive a simplified projection formula.**

(HINT: Consider writing $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$ where each $\vec{a}_i \in \mathbb{C}^m$ are orthonormal.)

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