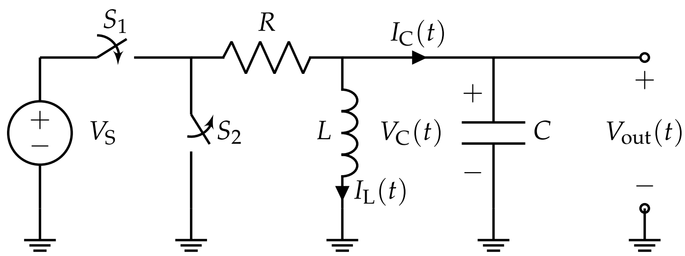
Discussion 6B

1. RLC Circuit with Vector Differential Equations

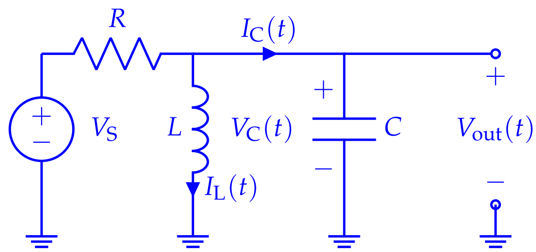
Consider the following circuit fed by a constant voltage source V_S.



The switch S_1, open for $t < 0$, closes at $t = 0$, and the switch S_2, closed for $t < 0$, opens at $t = 0$. Assume $V_C(0) = 0$ and $I_L(0) = 0$.

(a) **Derive a set of two differential equations, one for $I_L(t)$, the current through the inductor, and one for $V_C(t)$, the voltage across the capacitor.** Write your answer in terms of R, L, C, V_S, and constants.

Solution: The circuit appears as follows:



From Ohm's law for the resistor and KCL at the node with voltage V_C, we have:

$$\frac{V_S - V_C}{R} = I_L + I_C \tag{1}$$

Substituting in the I-V relationship for capacitors in the previous equation, we now have:

$$\frac{V_S - V_C}{R} = I_L + C\frac{\mathrm{d}V_C}{\mathrm{d}t} \tag{2}$$

$$\frac{V_S}{RC} - \frac{V_C}{RC} = \frac{I_L}{C} + \frac{\mathrm{d}V_C}{\mathrm{d}t} \tag{3}$$

$$\frac{V_S}{RC} - \frac{V_C}{RC} - \frac{I_L}{C} = \frac{\mathrm{d}V_C}{\mathrm{d}t} \tag{4}$$

Now, we can notice that $V_C = V_L$ as the inductor and capacitor are in parallel. From the inductor I-V relationship, we have:

$$L\frac{\mathrm{d}I_L}{\mathrm{d}t} = V_L = V_C \tag{5}$$

$$\frac{dI_L}{dt} = \frac{V_C}{L} \tag{6}$$

In summary, the two differential equations are as follows.

$$\frac{dV_C}{dt} = -\frac{V_C}{RC} - \frac{I_L}{C} + \frac{V_S}{RC} \tag{7}$$
$$\frac{dI_L}{dt} = \frac{V_C}{L} \tag{8}$$

(b) **Using your answers from the previous part, create a vector differential equation with the state vector being $\vec{x}(t) = \begin{bmatrix} V_C(t) \\ I_L(t) \end{bmatrix}$.** Write your answers in terms of R, L, C, V_S, and constants.

Solution: The previous part has the following differential equations.

$$\frac{dV_C}{dt} = -\frac{V_C}{RC} - \frac{I_C}{C} + \frac{V_S}{RC} \tag{9}$$
$$\frac{dI_L}{dt} = \frac{V_C}{L} \tag{10}$$

Stacking the above equations into matrix-vector form, we have

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix} V_S \tag{11}$$

(c) Regardless of your answer to the previous part, suppose the vector differential equation is given by

$$\frac{d}{dt}\vec{x}(t) = \underbrace{\begin{bmatrix} -4 & -6 \\ \frac{1}{2} & 0 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 4 \\ 0 \end{bmatrix}}_{\vec{b}} V_S \tag{12}$$

First, find the eigenvalues of the matrix A.

Solution: To find our eigenvalues, we use the method used in EECS 16A:

$$A\vec{v} = \lambda\vec{v} \tag{13}$$
$$A\vec{v} - \lambda\vec{v} = 0 \tag{14}$$
$$(A - \lambda I_2)\vec{v} = 0 \tag{15}$$
$$\det(A - \lambda I_2) = 0 \tag{16}$$
$$\det\left(\begin{bmatrix} -4 & -6 \\ \frac{1}{2} & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0 \tag{17}$$
$$\det\left(\begin{bmatrix} -4-\lambda & -6 \\ \frac{1}{2} & -\lambda \end{bmatrix}\right) = 0 \tag{18}$$
$$\lambda^2 + 4\lambda + 3 = 0 \tag{19}$$

Therefore our eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -1$.

(d) **Next, find the eigenvectors that will form your V basis.**

Solution: Recall, that the eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = -1$. We find the corresponding eigenvectors by plugging in our eigenvalues:
For $\lambda_1 = -3$:

$$(\begin{bmatrix} -4 & -6 \\ \frac{1}{2} & 0 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix})\vec{v_1} = 0 \tag{20}$$

$$(\begin{bmatrix} -1 & -6 \\ \frac{1}{2} & 3 \end{bmatrix})\vec{v_1} = 0 \tag{21}$$

From inspection or Gaussian Elimination, we find that $\vec{v_1} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$.

For $\lambda_1 = -1$:

$$(\begin{bmatrix} -4 & -6 \\ \frac{1}{2} & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix})\vec{v_2} = 0 \tag{22}$$

$$(\begin{bmatrix} -3 & -6 \\ \frac{1}{2} & 1 \end{bmatrix})\vec{v_2} = 0 \tag{23}$$

$$\tag{24}$$

From inspection or Gaussian Elimination, we find that $\vec{v_2} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

(e) **Now, in order to diagnolize the system, write A in terms of V, V^{-1}, and Λ.**

Solution: From Note 4, we know that:

$$\underbrace{\begin{bmatrix} -4 & -6 \\ \frac{1}{2} & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} -6 & -2 \\ 1 & 1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{3}{2} \end{bmatrix}}_{V^{-1}} \tag{25}$$

(f) **With $\vec{x}(0) = \vec{0}$, solve for $\vec{x}(t)$ and find the asymptotic/steady-state behavior as $t \to \infty$.** **Solution:** We can define $\vec{\tilde{x}}(t)$ to be the representation of $\vec{x}(t)$ in V-basis. In other words, $\vec{\tilde{x}}(t) = V^{-1}\vec{x}(t)$ and $\vec{x}(t) = V\vec{\tilde{x}}(t)$. Applying this, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = A\vec{x}(t) + \vec{b}V_S \tag{26}$$

$$V^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t)\right) = \underbrace{V^{-1}AV}_{\Lambda}\vec{\tilde{x}}(t) + V^{-1}\vec{b}V_S \tag{27}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{x}}(t) = \Lambda\vec{\tilde{x}}(t) + \underbrace{\begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix}}_{V^{-1}\vec{b}} V_S \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\tilde{x}}(t) = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}\vec{\tilde{x}}(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} V_S \tag{29}$$

The initial condition is $\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0) = V^{-1}\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We can solve these equations one row at a time, either by using substitution or the general first order differential equation solution. We use the latter approach. Solve the first row equation for $\widetilde{x}_1(t)$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}_1 = -3\widetilde{x}_1 - V_S \tag{30}$$

$$\implies \widetilde{x}_1(t) = -V_S e^{-3t}\int_0^t e^{3\theta}\,\mathrm{d}\theta \tag{31}$$

$$= -V_S e^{-3t}\frac{e^{3\theta}}{3}\Big|_{\theta=0}^{\theta=t} \tag{32}$$

$$= -V_S e^{-3t}\frac{e^{3t}-1}{3} \tag{33}$$

$$= V_S\frac{e^{-3t}-1}{3} \tag{34}$$

And now solve the second row equation for $\widetilde{x}_2(t)$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}_2 = -\widetilde{x}_2 + V_S \tag{35}$$

$$\implies \widetilde{x}_2(t) = V_S e^{-t}\int_0^t e^{\theta}\,\mathrm{d}\theta \tag{36}$$

$$= V_S e^{-t}(e^t - 1) \tag{37}$$

$$= V_S(1 - e^{-t}) \tag{38}$$

so

$$\vec{\widetilde{x}}(t) = V_S\begin{bmatrix}\frac{e^{-3t}-1}{3}\\ 1-e^{-t}\end{bmatrix} \tag{39}$$

Therefore,

$$\vec{x}(t) = V\vec{\widetilde{x}}(t) = V_S\begin{bmatrix}-6 & -2\\ 1 & 1\end{bmatrix}\begin{bmatrix}\frac{e^{-3t}-1}{3}\\ 1-e^{-t}\end{bmatrix} = V_S\begin{bmatrix}2\left(e^{-t}-e^{-3t}\right)\\ \frac{e^{-3t}-3e^{-t}+2}{3}\end{bmatrix} \tag{40}$$

Taking a limit as $t \to \infty$, we have

$$\lim_{t\to\infty}\vec{x}(t) = V_S\begin{bmatrix}0\\ \frac{2}{3}\end{bmatrix} \tag{41}$$

In particular, we can say that $\lim_{t\to\infty} V_C(t) = 0$, and that $\lim_{t\to\infty} I_L(t) = \frac{2}{3}V_S$.