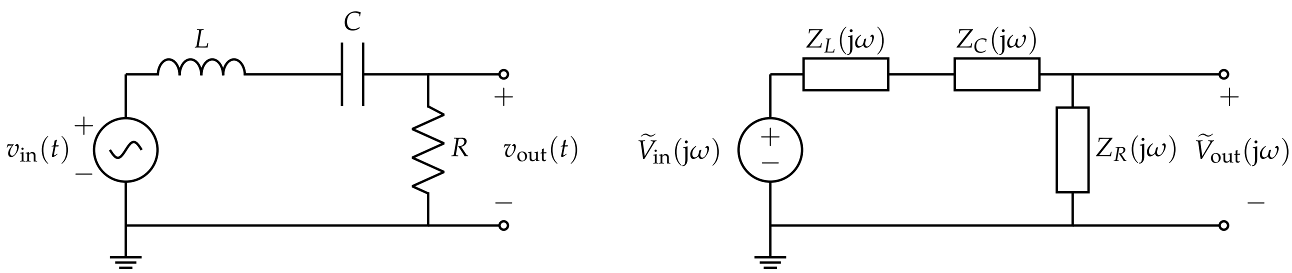
Discussion 5B

1. Band-Pass Filter

It is quite common to need to design a filter which selects only a narrow range of frequencies. One example is in WiFi radios, it is desirable to select only the 2.4GHz frequency containing your data, and reject information from other nearby cellular or bluetooth frequencies. This type of filter is called a band-pass filter; we will explore the design of this type of filter in this problem.



(a) Circuit in "time domain" **(b)** Circuit in "phasor domain"

(a) **Write down the transfer function** $H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}(\mathrm{j}\omega)}{\widetilde{V}_{\text{in}}(\mathrm{j}\omega)}$ **for this circuit as a rational function of** $\mathrm{j}\omega$ (make sure the numerator and denominator are both polynomials of $\mathrm{j}\omega$).

Solution: Using the same voltage divider rule we've used in the past, $\widetilde{V}_{\text{out}}(\mathrm{j}\omega)$ is:

$$
\begin{aligned}
\widetilde{V}_{\text{out}}(\mathrm{j}\omega) &= \widetilde{V}_{\text{in}}(\mathrm{j}\omega)\frac{Z_R}{Z_R + Z_L + Z_C} && (1)\\
&= \widetilde{V}_{\text{in}}\frac{R}{R + \mathrm{j}\omega L + \frac{1}{\mathrm{j}\omega C}} && (2)\\
\implies H(\mathrm{j}\omega) &= \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} && (3)\\
&= \frac{R}{R + \mathrm{j}\omega L + \frac{1}{\mathrm{j}\omega C}} && (4)\\
&= \frac{\mathrm{j}\omega RC}{\mathrm{j}\omega RC + (\mathrm{j}\omega)^2 LC + 1} && (5)\\
&= \frac{\mathrm{j}\omega RC}{(\mathrm{j}\omega)^2 LC + \mathrm{j}\omega RC + 1} && (6)
\end{aligned}
$$

(b) Consider the inductor, capacitor, and resistor connected in series. **Write down the impedance of the series RLC combination in the form** $Z_{RLC}(\mathrm{j}\omega) = A(\omega) + \mathrm{j}X(\omega)$**, where** $A(\omega)$ **and** $X(\omega)$ **are real valued functions of** ω**. At what frequency** ω_0 **does** $X(\omega_0) = 0$**?** (i.e. at what frequency is the impedance of the series combination of RLC purely real — meaning that the imaginary terms coming from the capacitor and inductor completely cancel each other. This is called the *resonant frequency*.) **Describe what the equivalent impedance of the inductor and capacitor is at resonance.**

Solution: Recall that the series impedance is the denominator of the voltage divider formula.

From the previous part, $Z_{RLC} = Z_R + Z_L + Z_C = R + \mathrm{j}\omega L + \frac{1}{\mathrm{j}\omega C} = R + \mathrm{j}(\omega L - \frac{1}{\omega C})$. Thus, $A(\omega) = R$ and $X(\omega) = \omega L - \frac{1}{\omega C}$.

Now, we can proceed to find ω_0.

$$X(\omega_0) = \omega_0 L - \frac{1}{\omega_0 C} = 0 \tag{7}$$

Multiplying both sides by ω_0:

$$\omega_0^2 L - \frac{1}{C} = 0 \tag{8}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}. \tag{9}$$

This expression for resonant frequency is very common and will show up often with RLC circuits!

At resonance, it turns out that the inductor and capacitor in series will be equivalent to a short circuit since their impedances will cancel out.

To show this, we can calculate the expression for the equivalent impedance:

$$Z_{LC} = \mathrm{j}\omega L + \frac{1}{\mathrm{j}\omega C} = \mathrm{j}(\omega L - \frac{1}{\omega C}) \tag{10}$$

This is the same expression that shows up when solving for the resonant frequency; the resonant frequency is the frequency that makes $Z_{LC} = 0$, which is equivalent to a short circuit!

(c) **Find the ratio of the magnitude of the phasors of inductor voltage to resistor voltage at resonance ($\omega = \omega_0$).** For a series resonance circuit, this is called the **quality factor** (Q) of the circuit and is an important metric to pay attention to (as we will see).

Solution: The voltage across each element can be found using a voltage divider corresponding to each element (we will avoid using the impedance expressions until later since there will be simplifications that can be made):

$$\widetilde{V}_L = \frac{Z_L}{Z_R + Z_L + Z_C}\widetilde{V}_{\text{in}} \tag{11}$$

$$\widetilde{V}_R = \frac{Z_R}{Z_R + Z_L + Z_C}\widetilde{V}_{\text{in}} \tag{12}$$

$$\tag{13}$$

We can notice that the ratio will be relatively simple (we use $\omega = \omega_0$ since we evaluate at resonance).

$$Q = \frac{|\widetilde{V}_L|}{|\widetilde{V}_R|} = \frac{|Z_L|}{|Z_R|} = \frac{|\mathrm{j}\omega_0 L|}{|R|} = \frac{\omega_0 L}{R} \tag{14}$$

(d) **Show that the transfer function found in part (a) can be written in the form**:

$$H(\mathrm{j}\omega) = \frac{1}{1 + \mathrm{j}Q(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})} \tag{15}$$

Solution: We start with the transfer function from part (a) and do some algebra to alter the expression's form (we make use of the fact that $C = \frac{1}{\omega_0^2 L}$):

$$H(\mathrm{j}\omega) = \frac{\mathrm{j}\omega RC}{(\mathrm{j}\omega)^2 LC + \mathrm{j}\omega RC + 1} \tag{16}$$

$$= \frac{1}{\mathrm{j}\omega\frac{L}{R} + 1 + \frac{1}{\mathrm{j}\omega RC}} \tag{17}$$

$$= \frac{1}{\mathrm{j}\omega\frac{Q}{\omega_0} + 1 + \frac{\omega_0^2 L}{\mathrm{j}\omega R}} \tag{18}$$

$$= \frac{1}{1 + \mathrm{j}Q\frac{\omega}{\omega_0} - \mathrm{j}Q\frac{\omega_0}{\omega}} \tag{19}$$

$$= \frac{1}{1 + \mathrm{j}Q(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega})} \tag{20}$$

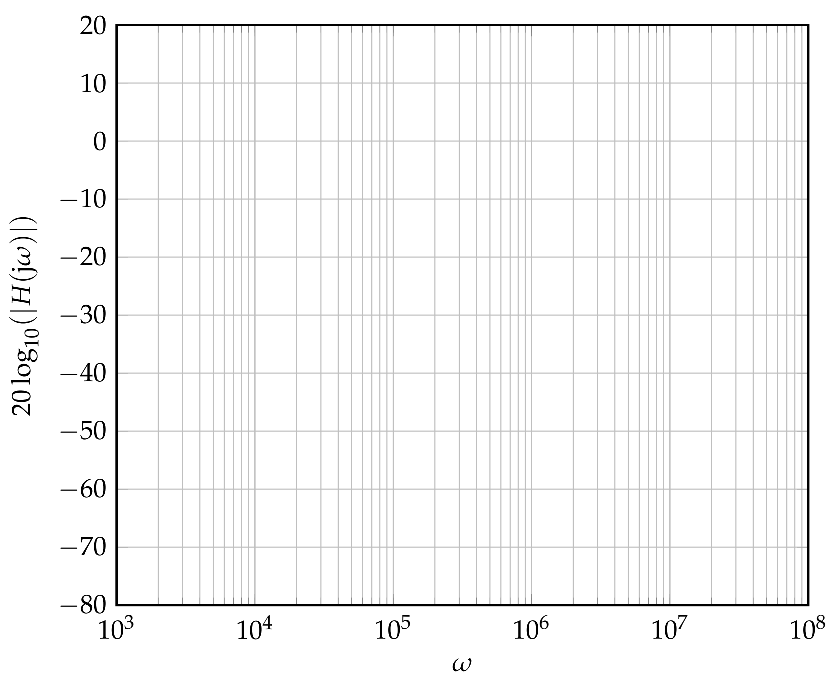
(e) With some derivation, we could show that if we have a high quality factor Q, we could find that the cutoff frequencies will be approximately:

$$\omega_c = \omega_0 \pm \frac{B}{2} \tag{21}$$

where B is the bandwidth and can be calculated as $B = \frac{\omega_0}{Q}$.

Suppose we have $R = 100\,\Omega$, $C = 10\,\mathrm{nF}$, $L = 10\,\mathrm{mH}$. **Find the cutoff frequencies, and draw a Bode plot for $|H(\mathrm{j}\omega)|$ on the provided plot.** *(HINT: Approximate the transfer function for $\omega < \omega_0$ and $\omega > \omega_0$ and combine this with the knowledge that $H(\mathrm{j}\omega_0) = 1$.)*

Plot of $|H(\mathrm{j}\omega)|_{dB}$ (for you to draw).



Solution: We can calculate the resonant frequency, quality factor, and bandwidth:

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(10^{-8})(10^{-2})}} = 10^5 \tag{22}$$

$$Q = \frac{\omega_0 L}{R} = \frac{(10^5)(10^{-2})}{100} = 10 \tag{23}$$

$$B = \frac{\omega_0}{Q} = \frac{10^5}{10} = 10^4 \tag{24}$$

With these values, we can approximate to have cutoff frequencies at $\omega_{c1} = \omega_0 - \frac{B}{2} = 10^5 - 0.5 \times 10^4 = 9.5 \times 10^4$ and $\omega_{c2} = \omega_0 + \frac{B}{2} = 10^5 + 0.5 \times 10^4 = 1.05 \times 10^5$. You may notice that these cutoff frequencies are very close to each other, which imply the presence of a very narrow range of frequencies that will be passed through without attenuation.

Now, we will look to try and draw the Bode plot for this transfer function. For $\omega < \omega_0$, we can approximate that $1 + jQ(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}) \approx -jQ\frac{\omega_0}{\omega}$ so the magnitude of the transfer function will be approximately:

$$|H(j\omega)| \approx \frac{1}{Q\frac{\omega_0}{\omega}} = \frac{\omega}{Q\omega_0} \tag{25}$$

This looks similar to a single zero transfer function (with $\omega_z = 0$), so we can expect a slope of $+20\frac{\text{dB}}{\text{dec}}$.

Also, at $\omega = \omega_0$, this transfer function approximation approaches $\frac{1}{Q} = \frac{1}{10}$, or -20dB.

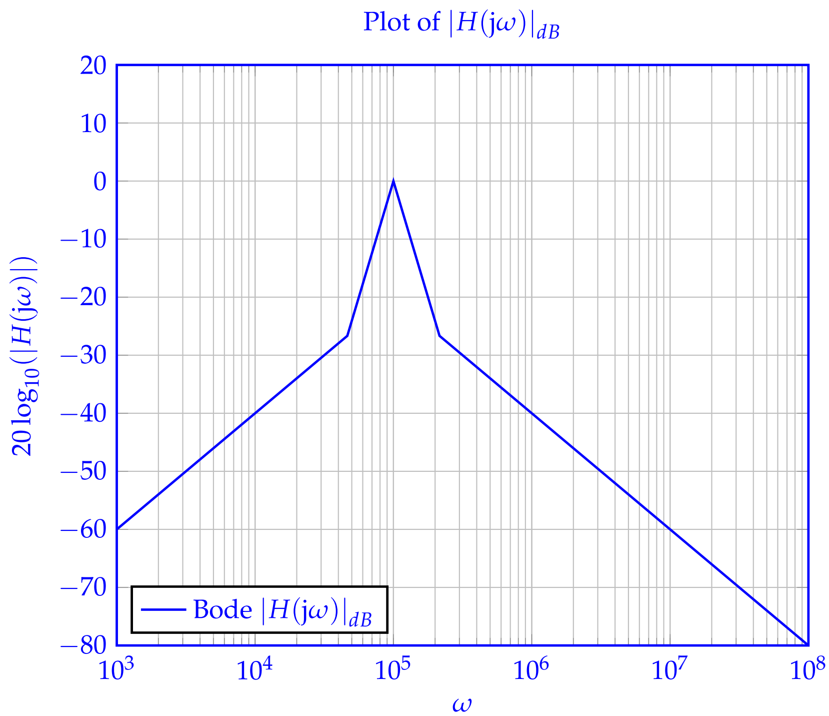
We can do similar analysis for $\omega > \omega_0$, where $1 + jQ(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}) \approx jQ\frac{\omega}{\omega_0}$.

$$|H(j\omega)| \approx \frac{1}{Q\frac{\omega}{\omega_0}} = \frac{\omega_0}{Q\omega} \tag{26}$$

This looks similar to a single pole transfer function (with $\omega_p = 0$), so we can expect a slope of $-20\frac{\text{dB}}{\text{dec}}$.

Also, at $\omega = \omega_0$, this transfer function approximation approaches $\frac{1}{Q} = \frac{1}{10}$, or -20dB (just as the other approximation).

However, we know that at $\omega = \omega_0$, the actual transfer function value is 1, or 0dB, so what we will observe is a peak for the transfer function Bode plot at the resonant frequency ω_0, with the height of the peak being around the 20dB difference between the asymptotic values of the two approximations around ω_0 and the actual transfer function value at ω_0. The Bode plot for this transfer function will look as follows (the peak of the plot does not have to be exact, but you should try to make it pass through the cutoff frequencies found earlier in the problem):



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