Discussion 5A

1. Poles and Zeros

For the last few discussions, you have worked with transfer functions and learned how to derive transfer functions for a variety of different circuits. In this discussion, we will learn about a very important tool for analyzing transfer functions called **Bode plots**. Bode plots are essentially plots of transfer functions over frequency (since transfer functions are complex valued, we will plot the magnitude and phase separately). When constructing these plots by hand, we will make use of approximations to simplify the process (since it is much faster to analyze without calculating the value of the transfer function at several different points, and our approximations will be relatively accurate anyways). These approximations make use of the **poles and zeros** of transfer functions, which we will describe now.

Suppose we write the transfer function in the following form:

$$H(j\omega) = K\frac{(j\omega)^{N_{z0}}(1+j\frac{\omega}{\omega_{z1}})(1+j\frac{\omega}{\omega_{z2}})\cdots(1+j\frac{\omega}{\omega_{zn}})}{(j\omega)^{N_{p0}}(1+j\frac{\omega}{\omega_{p1}})(1+j\frac{\omega}{\omega_{p2}})\cdots(1+j\frac{\omega}{\omega_{pm}})} \tag{1}$$

Each frequency ω_z is called a **zero frequency** and each frequency ω_p is called a **pole frequency**. The presence of a factor $j\omega$ in the numerator of the transfer function indicates a zero frequency at 0, and the presence of a factor $j\omega$ in the denominator of the transfer function indicates a pole frequency at 0.

For the horizontal axis for Bode plots, we plot frequency ω in log scale ($x = \log(\omega)$).

For Bode plots of the transfer function magnitude, we measure the value in decibel scale:

$$y = |H(j\omega)|_{\text{dB}} = 20\log(|H(j\omega)|) \tag{2}$$

We can use the properties of logarithms to see how our expression in terms of poles and zeros works in decibel scale:

$$\begin{aligned}
&20\log|K\frac{(j\omega)^{N_{z0}}(1+j\frac{\omega}{\omega_{z1}})(1+j\frac{\omega}{\omega_{z2}})\cdots(1+j\frac{\omega}{\omega_{zn}})}{(j\omega)^{N_{p0}}(1+j\frac{\omega}{\omega_{p1}})(1+j\frac{\omega}{\omega_{p2}})\cdots(1+j\frac{\omega}{\omega_{pm}})}| \\
&= 20\log|K| + 20N_{z0}\log|j\omega| + 20\log|1+j\frac{\omega}{\omega_{z1}}| + \ldots + 20\log|1+j\frac{\omega}{\omega_{zn}}| \\
&+ 20N_{p0}\log|\frac{1}{j\omega}| + 20\log|\frac{1}{1+j\frac{\omega}{\omega_{p1}}}| + \ldots + 20\log|\frac{1}{1+j\frac{\omega}{\omega_{pm}}}|
\end{aligned}$$

This is similar with phase, where the overall phase is the sum of each factor's phase (by properties of complex numbers):

$$\begin{aligned}
&\measuredangle K\frac{(j\omega)^{N_{z0}}(1+j\frac{\omega}{\omega_{z1}})(1+j\frac{\omega}{\omega_{z2}})\cdots(1+j\frac{\omega}{\omega_{zn}})}{(j\omega)^{N_{p0}}(1+j\frac{\omega}{\omega_{p1}})(1+j\frac{\omega}{\omega_{p2}})\cdots(1+j\frac{\omega}{\omega_{pm}})} \\
&= \measuredangle K + N_{z0}\measuredangle(j\omega) + \measuredangle(1+j\frac{\omega}{\omega_{z1}}) + \ldots + \measuredangle(1+j\frac{\omega}{\omega_{zn}}) + N_{p0}\measuredangle(\frac{1}{j\omega}) + \measuredangle(\frac{1}{1+j\frac{\omega}{\omega_{p1}}}) + \ldots + \measuredangle(\frac{1}{1+j\frac{\omega}{\omega_{pm}}})
\end{aligned}$$

What this tells us is that if we can analyze poles and zeros independently, we can combine them using addition to plot more complex transfer functions!

To start out, we will plot the Bode plot of a single pole:

$$H(\mathrm{j}\omega) = \frac{1}{1+\mathrm{j}\frac{\omega}{\omega_p}} \tag{3}$$

(a) Find and simplify expressions for $|H(\mathrm{j}\omega)|_{\mathrm{dB}}$ and $\measuredangle H(\mathrm{j}\omega)$ as much as possible.

Solution:

$$|H(\mathrm{j}\omega)|_{\mathrm{dB}} = 20\log(|H(\mathrm{j}\omega)|) = 20\log\left(\frac{1}{\sqrt{1+\frac{\omega^2}{\omega_p^2}}}\right) = -20\log\left(\sqrt{1+\frac{\omega^2}{\omega_p^2}}\right) \tag{4}$$

$$\measuredangle H(\mathrm{j}\omega) = \measuredangle(\frac{1}{1+\mathrm{j}\frac{\omega}{\omega_p}}) = -\measuredangle(1+\mathrm{j}\frac{\omega}{\omega_p}) = -\arctan\left(\frac{\omega}{\omega_p}\right) \tag{5}$$

(b) Approximate $|H(\mathrm{j}\omega)|_{\mathrm{dB}}$ for $\omega < \omega_p$ and $\omega > \omega_p$. What is the slope of each section (in units of $\frac{\mathrm{dB}}{\mathrm{dec}}$)?

Solution: For $\omega < \omega_p$, we know that $\frac{\omega^2}{\omega_p^2} < 1$ so we will approximate that $1 + \frac{\omega^2}{\omega_p^2} \approx 1$.

$$|H(\mathrm{j}\omega)|_{\mathrm{dB}} = -20\log\left(\sqrt{1+\frac{\omega^2}{\omega_p^2}}\right) \approx -20\log(1) = 0\mathrm{dB} \tag{6}$$

The slope in this case is $0\frac{\mathrm{dB}}{\mathrm{dec}}$. For $\omega > \omega_p$, we know that $\frac{\omega^2}{\omega_p^2} > 1$ so we will approximate that $1 + \frac{\omega^2}{\omega_p^2} \approx \frac{\omega^2}{\omega_p^2}$.

$$|H(\mathrm{j}\omega)|_{\mathrm{dB}} = -20\log\left(\sqrt{1+\frac{\omega^2}{\omega_p^2}}\right) \approx -20\log\left(\frac{\omega}{\omega_p}\right) = -20\log(\omega) + 20\log(\omega_p) \tag{7}$$

Since the horizontal axis for our Bode plots is $x = \log(\omega)$, this looks like the line $-20x + 20\log(\omega_p)$, which has a slope of $-20\frac{\mathrm{dB}}{\mathrm{dec}}$.

(c) Approximate $\measuredangle H(\mathrm{j}\omega)$ for $\omega < \frac{1}{10}\omega_p$ and $\omega > 10\omega_p$. Calculate the exact value at $\omega = \omega_p$.

Solution: For $\omega < \frac{1}{10}\omega_p$, we can approximate that $\frac{\omega}{\omega_p} \approx 0$.

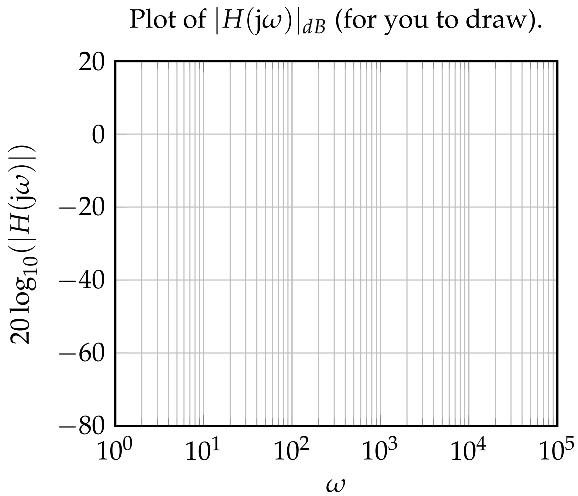
$$\measuredangle H(\mathrm{j}\omega) = -\arctan\left(\frac{\omega}{\omega_p}\right) \approx -\arctan(0) = 0 \tag{8}$$

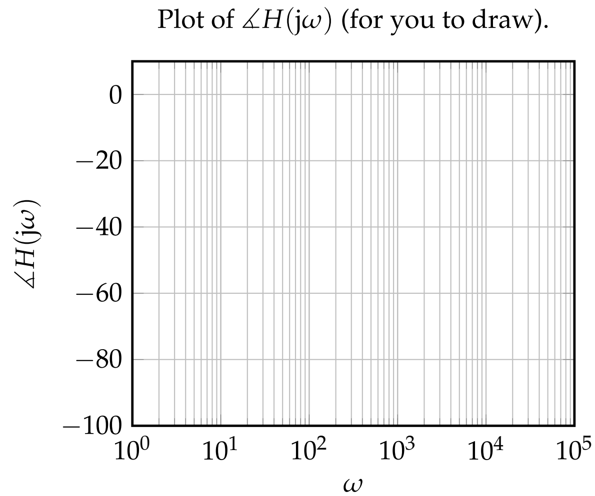
For $\omega > 10\omega_p$, we can approximate that $\frac{\omega}{\omega_p} \approx \infty$ (this may not make much sense, but for the arctan function, once the argument becomes high, the value asymptotes to what it would be if the argument was ∞).

$$\measuredangle H(\mathrm{j}\omega) = -\arctan\left(\frac{\omega}{\omega_p}\right) \approx -\arctan(\infty) = -90^\circ \tag{9}$$

For $\omega = \omega_p$, $\frac{\omega}{\omega_p} = 1$.

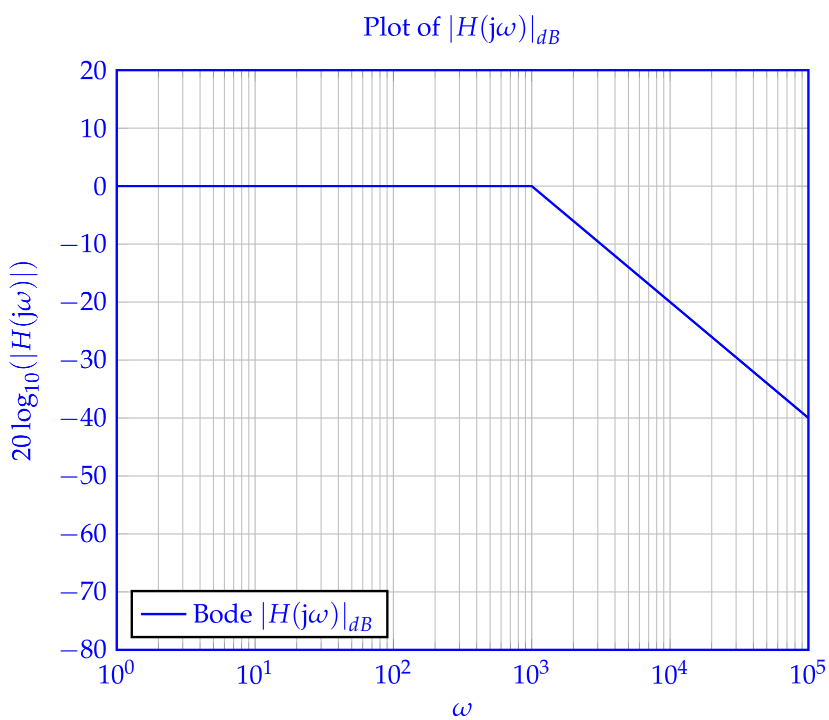
$$\measuredangle H(\mathrm{j}\omega) = -\arctan\left(\frac{\omega}{\omega_p}\right) = -\arctan(1) = -45^\circ \tag{10}$$

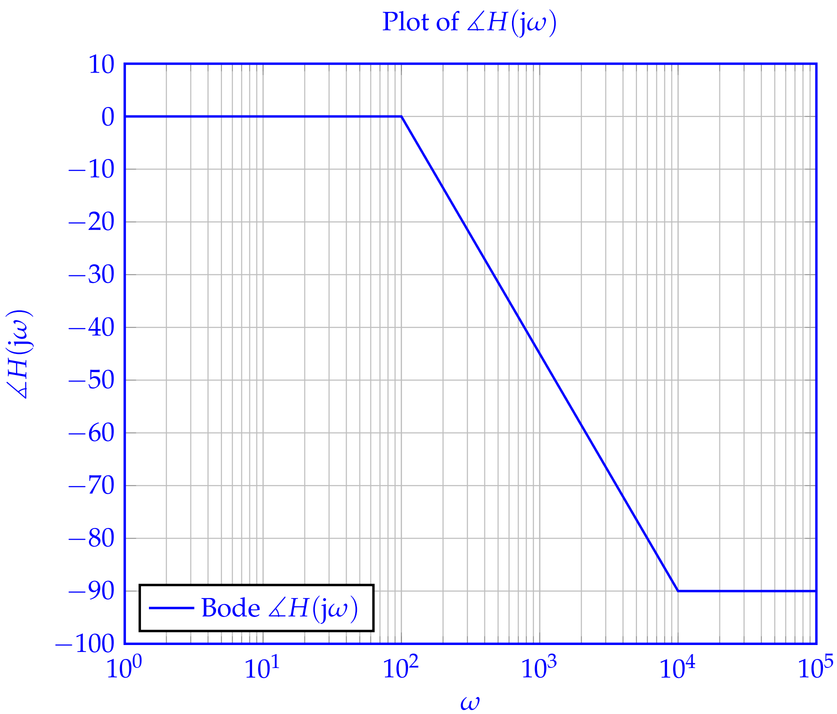




(d) Draw the Bode plots for this single pole transfer function on the below plots. You can assume $\omega_p = 10^3 \frac{\text{rad}}{\text{s}}$. *(HINT: Use the same approximation ranges as provided in the previous problems and only draw linear segments. For the phase plot, make the plot linear and pass through the $\omega = \omega_p$ value for the range $\frac{1}{10}\omega_p < \omega < 10\omega_p$.)*

Solution: We can draw lines as advised by the hint and our values from the previous problem parts.





From our plots, we should be able to notice that for the magnitude plot, the pole does not contribute (0dB) until $\omega = \omega_p$, at which point the pole cause the slope to decrease by $20\frac{\text{dB}}{\text{dec}}$. For the phase plot, the pole causes the phase to decrease linearly by 90° from $\frac{1}{10}\omega_p$ to $10\omega_p$.

If there is a pole at $\omega_p = 0$, the $-20\frac{\text{dB}}{\text{dec}}$ slope and $-90°$ phase are applied immediately (since you are always past $\omega_p = 0$ in log scale).

Now, we will do the same for a single zero:

$$H(\mathrm{j}\omega) = 1 + \mathrm{j}\frac{\omega}{\omega_z} \tag{11}$$

(e) Find and simplify expressions for $|H(\mathrm{j}\omega)|_{\mathrm{dB}}$ and $\measuredangle H(\mathrm{j}\omega)$ as much as possible.

Solution:

$$|H(\mathrm{j}\omega)|_{\mathrm{dB}} = 20\log(|H(\mathrm{j}\omega)|) = 20\log\left(\sqrt{1+\frac{\omega^2}{\omega_z^2}}\right) \tag{12}$$

$$\measuredangle H(\mathrm{j}\omega) = \measuredangle(1+\mathrm{j}\frac{\omega}{\omega_z}) = \arctan\left(\frac{\omega}{\omega_z}\right) \tag{13}$$

(f) Approximate $|H(\mathrm{j}\omega)|_{\mathrm{dB}}$ for $\omega < \omega_z$ and $\omega > \omega_z$. What is the slope of each section (in units of $\frac{\mathrm{dB}}{\mathrm{dec}}$)?

Solution: For $\omega < \omega_z$, we know that $\frac{\omega^2}{\omega_z^2} < 1$ so we will approximate that $1 + \frac{\omega^2}{\omega_z^2} \approx 1$.

$$|H(\mathrm{j}\omega)|_{\mathrm{dB}} = 20\log\left(\sqrt{1+\frac{\omega^2}{\omega_z^2}}\right) \approx 20\log(1) = 0\mathrm{dB} \tag{14}$$

The slope in this case is $0\frac{\mathrm{dB}}{\mathrm{dec}}$. For $\omega > \omega_z$, we know that $\frac{\omega^2}{\omega_z^2} > 1$ so we will approximate that $1 + \frac{\omega^2}{\omega_z^2} \approx \frac{\omega^2}{\omega_z^2}$.

$$|H(\mathrm{j}\omega)|_{\mathrm{dB}} = 20\log\left(\sqrt{1+\frac{\omega^2}{\omega_z^2}}\right) \approx 20\log\left(\frac{\omega}{\omega_z}\right) = 20\log(\omega) - 20\log(\omega_z) \tag{15}$$

Since the horizontal axis for our Bode plots is $x = \log(\omega)$, this looks like the line $20x - 20\log(\omega_z)$, which has a slope of $+20\frac{\mathrm{dB}}{\mathrm{dec}}$.

(g) Approximate $\measuredangle H(\mathrm{j}\omega)$ for $\omega < \frac{1}{10}\omega_z$ and $\omega > 10\omega_z$. Calculate the exact value at $\omega = \omega_z$.

Solution: For $\omega < \frac{1}{10}\omega_z$, we can approximate that $\frac{\omega}{\omega_z} \approx 0$.

$$\measuredangle H(\mathrm{j}\omega) = \arctan\left(\frac{\omega}{\omega_z}\right) \approx \arctan(0) = 0 \tag{16}$$

For $\omega > 10\omega_z$, we can approximate that $\frac{\omega}{\omega_z} \approx \infty$ (this may not make much sense, but for the arctan function, once the argument becomes high, the value asymptotes to what it would be if the argument was ∞).

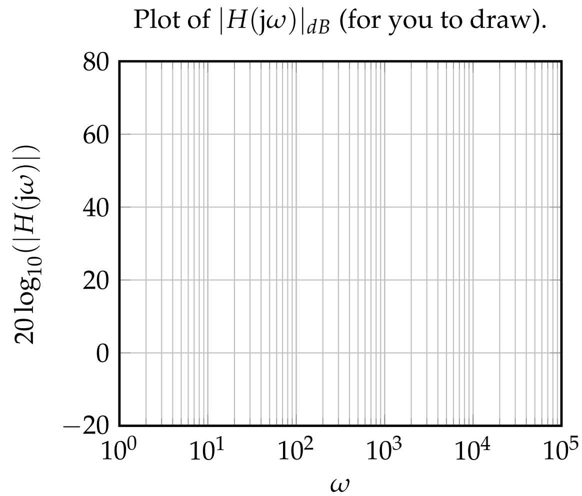
$$\measuredangle H(\mathrm{j}\omega) = \arctan\left(\frac{\omega}{\omega_z}\right) \approx \arctan(\infty) = 90^\circ \tag{17}$$

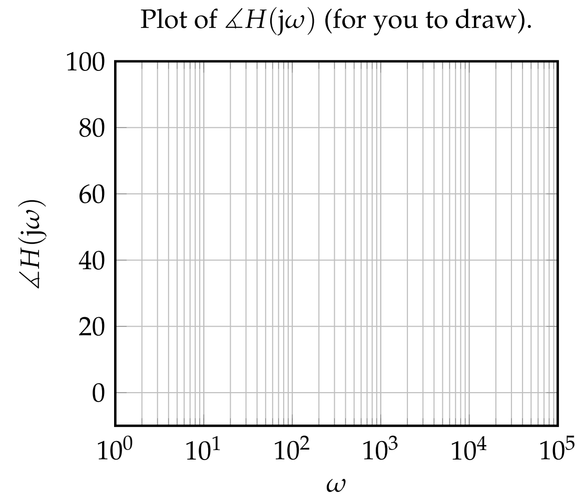
For $\omega = \omega_z$, $\frac{\omega}{\omega_z} = 1$.

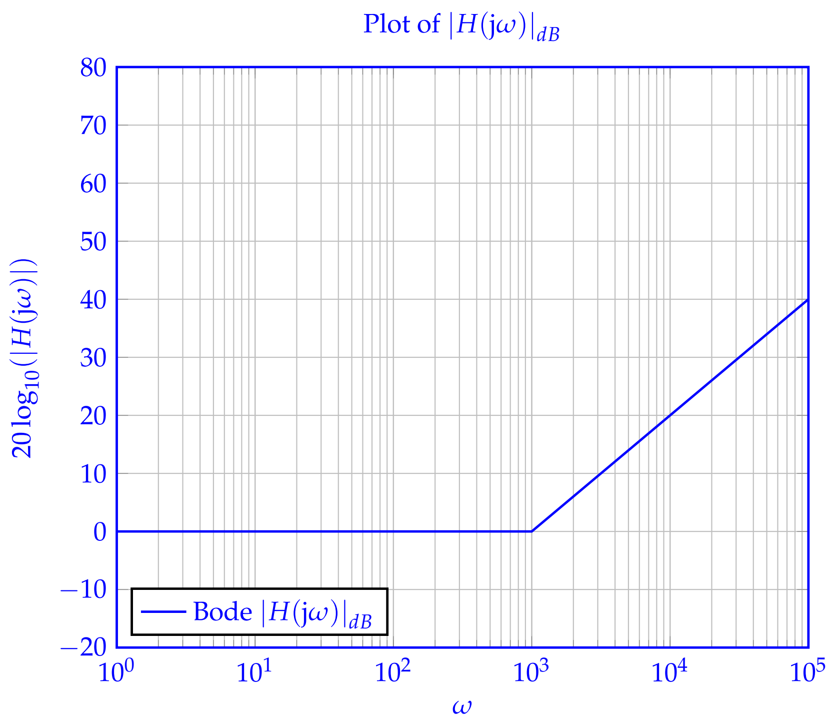
$$\measuredangle H(\mathrm{j}\omega) = \arctan\left(\frac{\omega}{\omega_z}\right) = \arctan(1) = 45^\circ \tag{18}$$

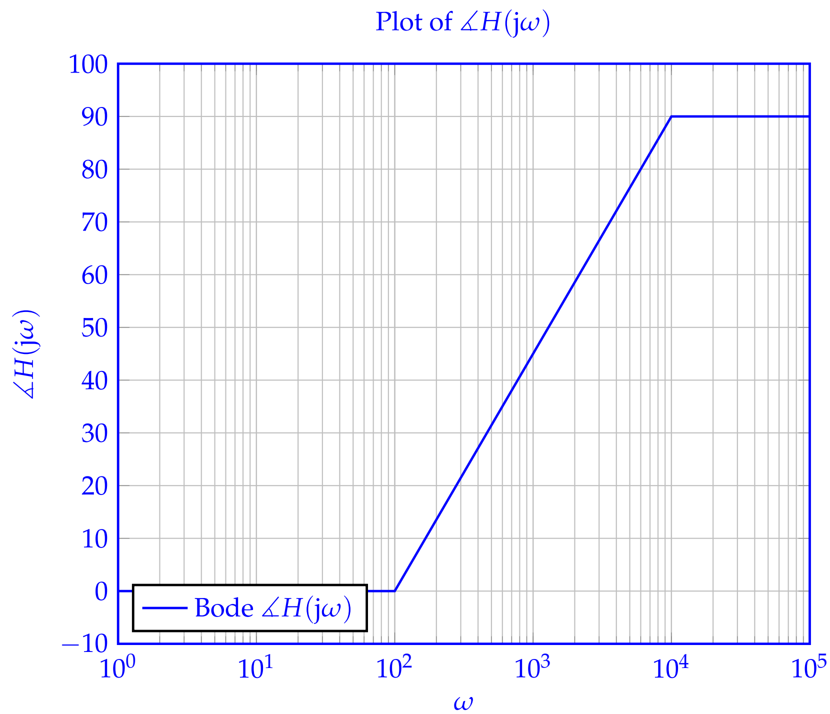
(h) Draw the Bode plots for this single pole transfer function on the below plots. You can assume $\omega_z = 10^3\,\frac{\mathrm{rad}}{\mathrm{s}}$. *(HINT: Use the same approximation ranges as provided in the previous problems and only draw linear segments. For the phase plot, make the plot linear and pass through the $\omega = \omega_z$ value for the range $\frac{1}{10}\omega_z < \omega < 10\omega_z$.)*

Solution: We can draw lines as advised by the hint and our values from the previous problem parts.


Plot of $|H(j\omega)|_{dB}$ (for you to draw).
80
60
40
20
0
−20
$20\log_{10}(|H(j\omega)|)$
10^0 10^1 10^2 10^3 10^4 10^5
ω


Plot of $\measuredangle H(j\omega)$ (for you to draw).
100
80
60
40
20
0
$\measuredangle H(j\omega)$
10^0 10^1 10^2 10^3 10^4 10^5
ω


Plot of $|H(j\omega)|_{dB}$
80
70
60
50
40
30
20
10
0
−10
−20
$20\log_{10}(|H(j\omega)|)$
Bode $|H(j\omega)|_{dB}$
10^0 10^1 10^2 10^3 10^4 10^5
ω



From our plots, we should be able to notice that for the magnitude plot, the zero does not contribute (0dB) until $\omega = \omega_z$, at which point the zero cause the slope to increase by $20\frac{\text{dB}}{\text{dec}}$. For the phase plot, the zero causes the phase to increase linearly by $90°$ from $\frac{1}{10}\omega_z$ to $10\omega_z$.

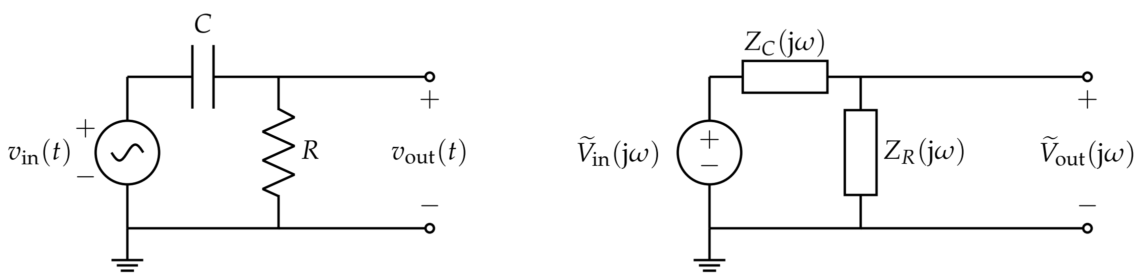
If there is a zero at $\omega_z = 0$, the $+20\frac{\text{dB}}{\text{dec}}$ slope and $+90°$ phase are applied immediately (since you are always past $\omega_z = 0$ in log scale).

2. Bode Plots Practice

With our knowledge of poles and zeros, let's plot some of the transfer functions we derived in previous discussions!

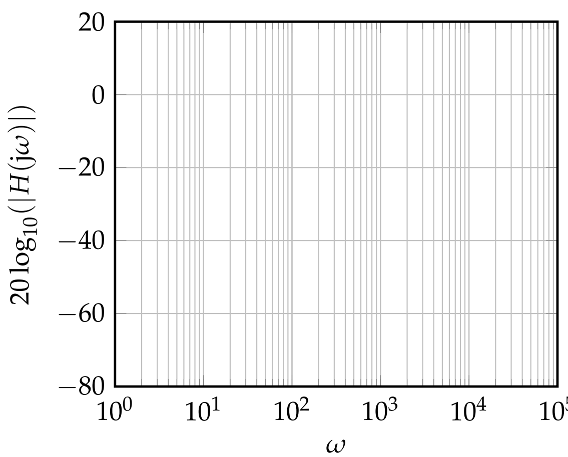
For all of the following parts, identify the poles and zeros, and plot the Bode plots of the transfer functions on the provided plots.

(a) **RC circuit** ($R = 1\,\mathrm{k}\Omega$, $C = 1\,\mu\mathrm{F}$):

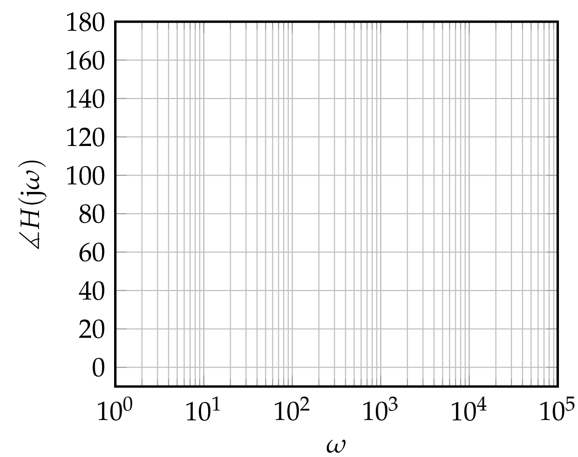


(a) Circuit in "time domain" **(b)** Circuit in "phasor domain"

Plot of $|H(\mathrm{j}\omega)|_{dB}$ (for you to draw).



Plot of $\measuredangle H(\mathrm{j}\omega)$ (for you to draw).



We found that the transfer function for this circuit was:

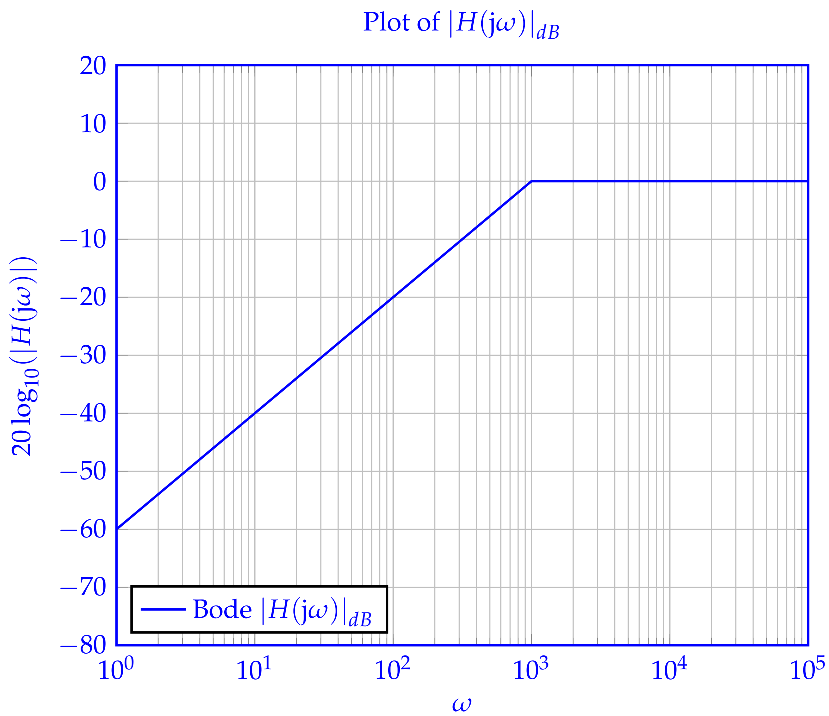
$$H(\mathrm{j}\omega) = \frac{\mathrm{j}\frac{\omega}{10^3}}{1 + \mathrm{j}\frac{\omega}{10^3}} \tag{19}$$

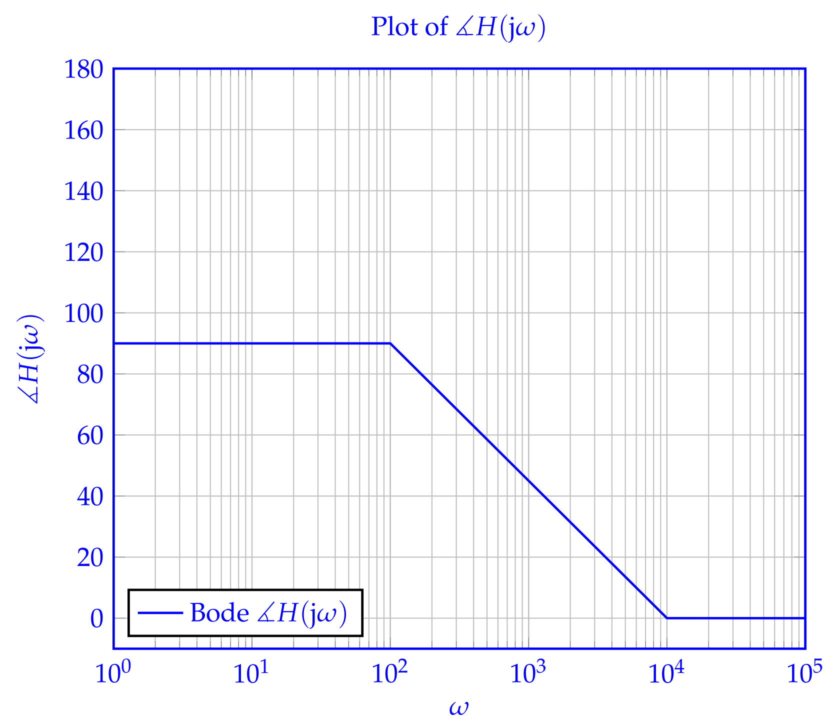
Solution: This transfer function has a zero at $\omega_z = 0$ and a pole at $\omega_p = 10^3$.

The magnitude plot will start off with a slope of $+20\frac{\mathrm{dB}}{\mathrm{dec}}$ due to $\omega_z = 0$, and at $\omega_p = 10^3$, the slope will decrease to $0\frac{\mathrm{dB}}{\mathrm{dec}}$, at which point the magnitude should be $20\log_{10}(|H(\mathrm{j}\infty)|) = 20\log_{10}(1) = 0\,\mathrm{dB}$.

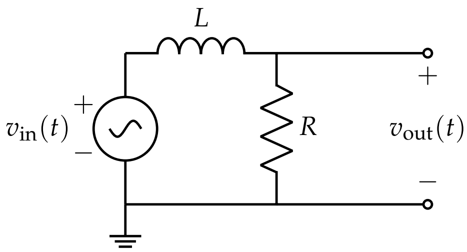
The phase plot will start off at $+90^\circ$ due to $\omega_z = 0$ and will decrease linearly in the Bode plot from 90° to 0 from 10^2 to 10^4 due to $\omega_p = 10^3$.

The magnitude and phase plots of $H(\mathrm{j}\omega)$ are shown below.

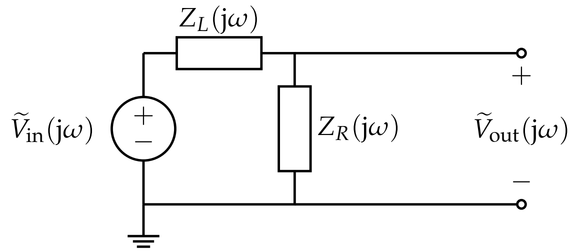

Plot of $|H(j\omega)|_{dB}$
$20\log_{10}(|H(j\omega)|)$
20
10
0
−10
−20
−30
−40
−50
−60
−70
−80
10^0
10^1
10^2
10^3
10^4
10^5
ω
Bode $|H(j\omega)|_{dB}$


Plot of $\measuredangle H(j\omega)$
$\measuredangle H(j\omega)$
180
160
140
120
100
80
60
40
20
0
10^0
10^1
10^2
10^3
10^4
10^5
ω
Bode $\measuredangle H(j\omega)$

(b) **LR circuit** ($L = 5\,\text{H}$, $R = 500\,\Omega$):

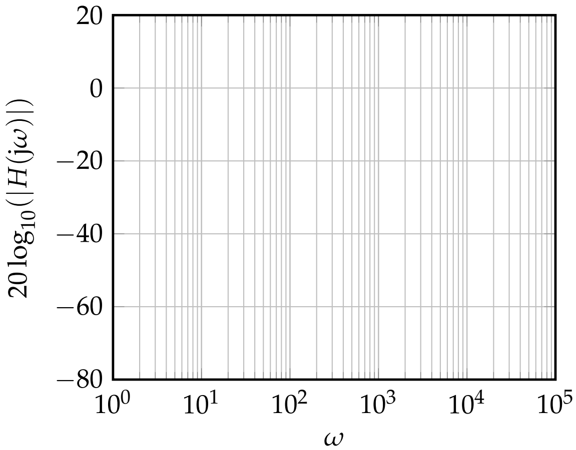


(a) Circuit in "time domain"

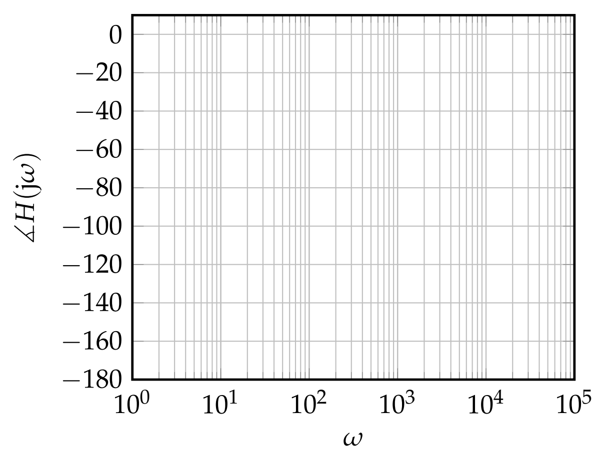


(b) Circuit in "phasor domain"

Plot of $|H(\text{j}\omega)|_{dB}$ (for you to draw).



Plot of $\measuredangle H(\text{j}\omega)$ (for you to draw).



We found that the transfer function for this circuit was:

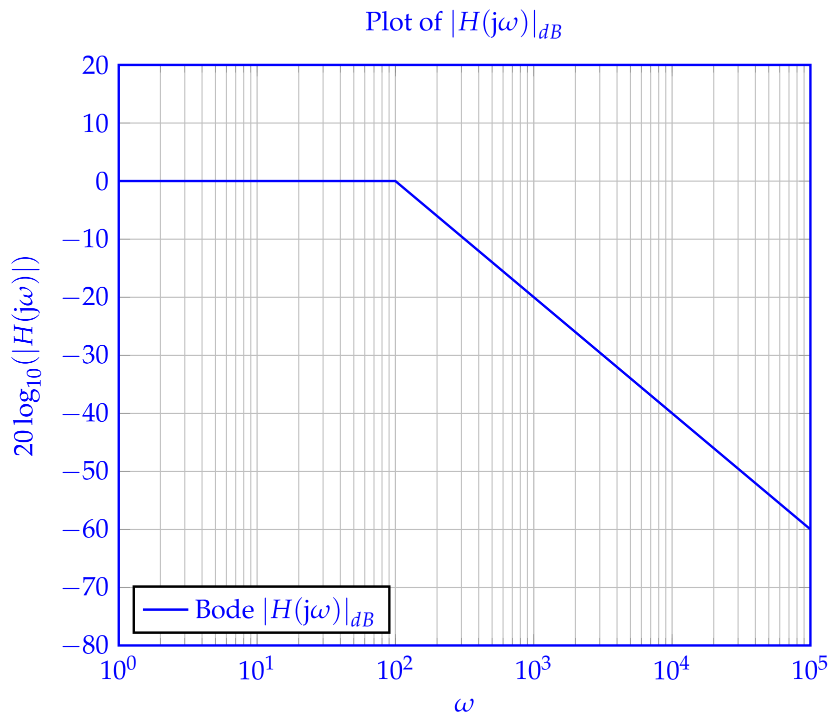
$$H(\text{j}\omega) = \frac{1}{1 + \text{j}\frac{\omega}{10^2}} \tag{20}$$

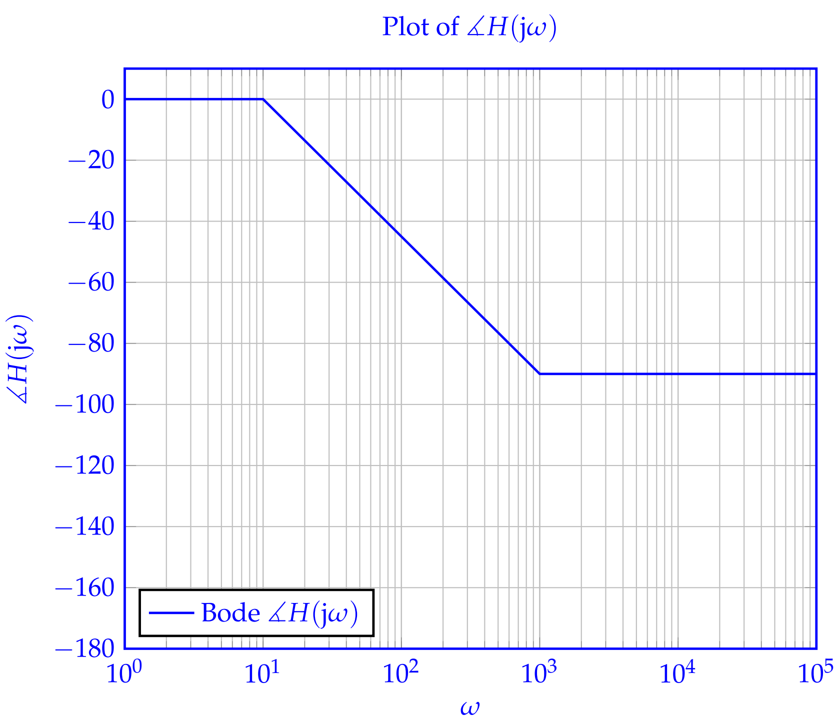
Solution: This transfer function has a pole at $\omega_p = 10^2$.

The magnitude plot will start off at $20\log_{10}(|H(\text{j}0)|) = 20\log_{10}(1) = 0\,\text{dB}$ and at $\omega_p = 10^2$, the slope will decrease to $-20\frac{\text{dB}}{\text{dec}}$.

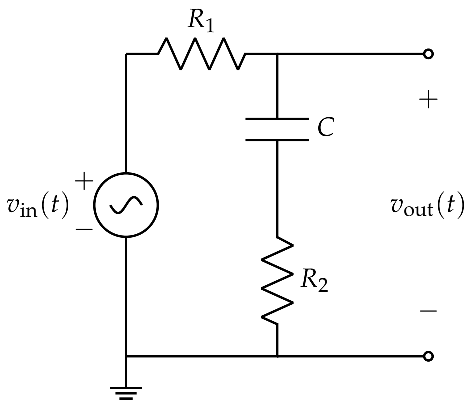
The phase plot will start off at 0 and will decrease linearly in the Bode plot from 0 to $-90°$ from 10^1 to 10^3 due to $\omega_p = 10^2$.

The magnitude and phase plots of $H(\text{j}\omega)$ are shown below.

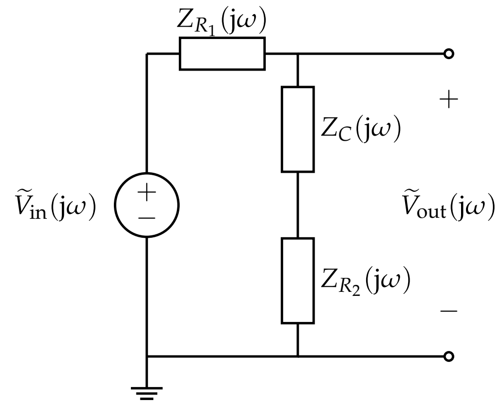

Plot of $|H(j\omega)|_{dB}$
20
10
0
−10
−20
−30
−40
−50
−60
−70
−80
$20\log_{10}(|H(j\omega)|)$
Bode $|H(j\omega)|_{dB}$
10^0
10^1
10^2
10^3
10^4
10^5
ω


Plot of $\measuredangle H(j\omega)$
0
−20
−40
−60
−80
−100
−120
−140
−160
−180
$\measuredangle H(j\omega)$
Bode $\measuredangle H(j\omega)$
10^0
10^1
10^2
10^3
10^4
10^5
ω

(c) **RCR circuit** ($R_1 = 9\,\mathrm{k\Omega}$, $R_2 = 1\,\mathrm{k\Omega}$, $C = 1\,\mathrm{\mu F}$):

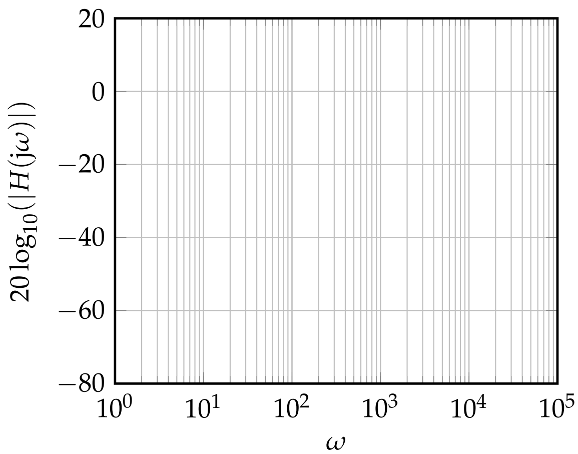


(a) Circuit in "time domain"

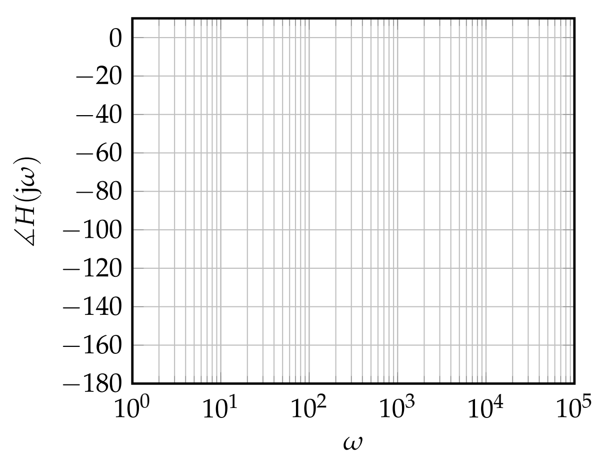


(b) Circuit in "phasor domain"

Plot of $|H(\mathrm{j}\omega)|_{dB}$ (for you to draw).



Plot of $\measuredangle H(\mathrm{j}\omega)$ (for you to draw).



We found that the transfer function for this circuit was:

$$H(\mathrm{j}\omega) = \frac{1 + \mathrm{j}\frac{\omega}{10^3}}{1 + \mathrm{j}\frac{\omega}{10^2}} \tag{21}$$

Solution: This transfer function is more complicated as it does not fit the basic low pass or high pass filter models we are used to so this is where using poles and zeros is very helpful.

This transfer function has a pole at $\omega_p = 10^2$ and a zero at $\omega_z = 10^3$.

The magnitude plot will start off at $20\log_{10}(|H(\mathrm{j}0)|) = 20\log_{10}(1) = 0\,\mathrm{dB}$.

At $\omega_p = 10^2$, the slope will decrease to $-20\frac{\mathrm{dB}}{\mathrm{dec}}$.

At $\omega_z = 10^3$, the slope will increase to $0\frac{\mathrm{dB}}{\mathrm{dec}}$, at which point the magnitude should be $20\log_{10}(|H(\mathrm{j}\infty)|) = 20\log_{10}(\frac{1}{10}) = -20\,\mathrm{dB}$.

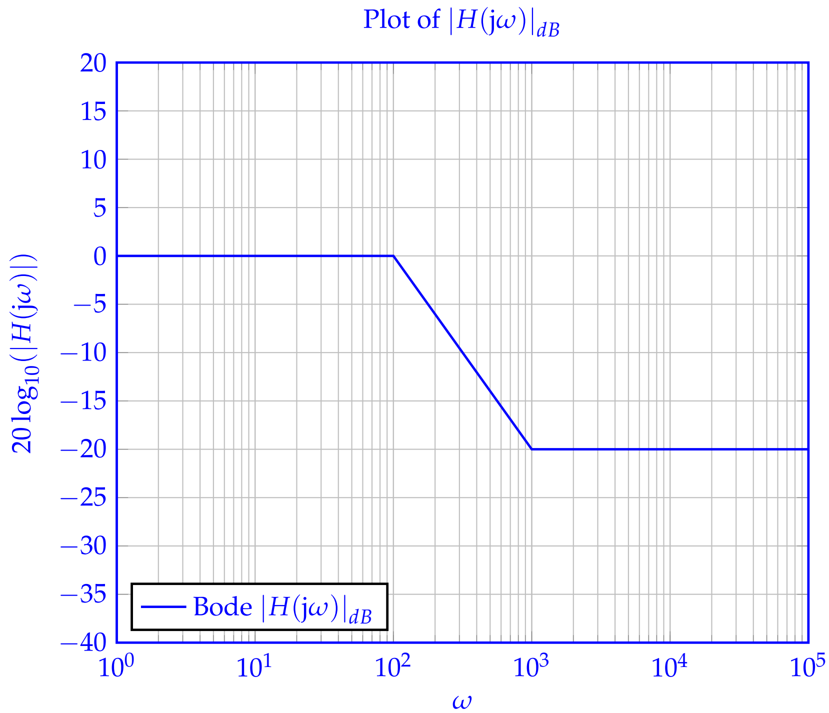
The phase plot will start off at 0.

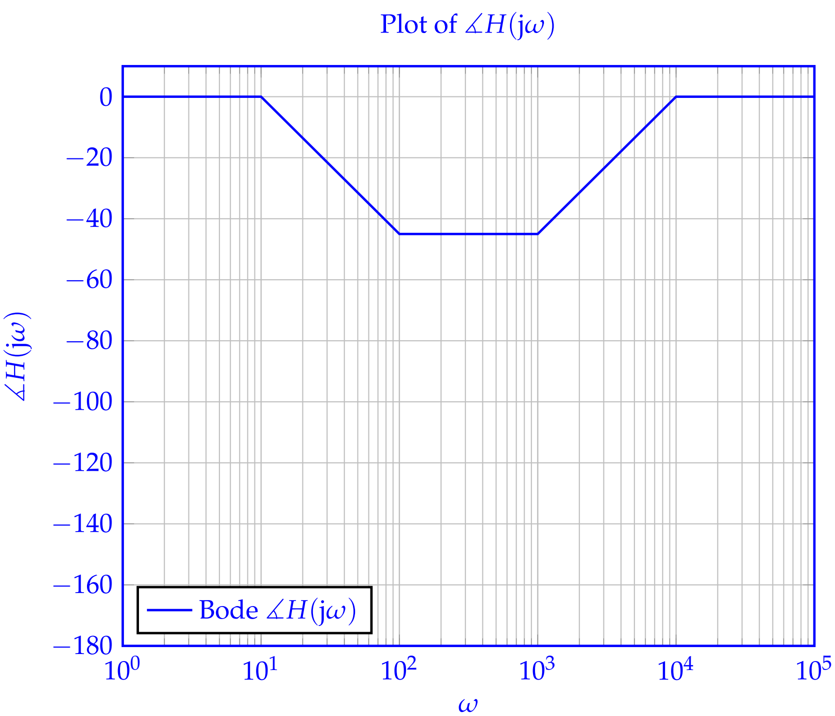
From 10^1 to 10^2, the phase will decrease linearly in the Bode plot from 0 to $-45°$ due to $\omega_p = 10^2$.

From 10^2 to 10^3, both the pole and zero change the phase and cancel each other's effects out so the phase stays constant at $-45°$.

From 10^3 to 10^4, the phase will increase linearly in the Bode plot from $-45°$ to 0 due to $\omega_z = 10^3$.

The magnitude and phase plots of $H(j\omega)$ are shown below.





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