Discussion 5A

1. Poles and Zeros

For the last few discussions, you have worked with transfer functions and learned how to derive transfer functions for a variety of different circuits. In this discussion, we will learn about a very important tool for analyzing transfer functions called **Bode plots**. Bode plots are essentially plots of transfer functions over frequency (since transfer functions are complex valued, we will plot the magnitude and phase separately). When constructing these plots by hand, we will make use of approximations to simplify the process (since it is much faster to analyze without calculating the value of the transfer function at several different points, and our approximations will be relatively accurate anyways). These approximations make use of the **poles and zeros** of transfer functions, which we will describe now.

Suppose we write the transfer function in the following form:

$$H(\mathrm{j}\omega) = K\frac{(\mathrm{j}\omega)^{N_{z0}}(1+\mathrm{j}\frac{\omega}{\omega_{z1}})(1+\mathrm{j}\frac{\omega}{\omega_{z2}})\cdots(1+\mathrm{j}\frac{\omega}{\omega_{zn}})}{(\mathrm{j}\omega)^{N_{p0}}(1+\mathrm{j}\frac{\omega}{\omega_{p1}})(1+\mathrm{j}\frac{\omega}{\omega_{p2}})\cdots(1+\mathrm{j}\frac{\omega}{\omega_{pm}})} \tag{1}$$

Each frequency ω_z is called a **zero frequency** and each frequency ω_p is called a **pole frequency**. The presence of a factor $\mathrm{j}\omega$ in the numerator of the transfer function indicates a zero frequency at 0, and the presence of a factor $\mathrm{j}\omega$ in the denominator of the transfer function indicates a pole frequency at 0.

For the horizontal axis for Bode plots, we plot frequency ω in log scale ($x = \log(\omega)$).

For Bode plots of the transfer function magnitude, we measure the value in decibel scale:

$$y = |H(\mathrm{j}\omega)|_{\mathrm{dB}} = 20\log(|H(\mathrm{j}\omega)|) \tag{2}$$

We can use the properties of logarithms to see how our expression in terms of poles and zeros works in decibel scale:

$$\begin{aligned}
&20\log|K\frac{(\mathrm{j}\omega)^{N_{z0}}(1+\mathrm{j}\frac{\omega}{\omega_{z1}})(1+\mathrm{j}\frac{\omega}{\omega_{z2}})\cdots(1+\mathrm{j}\frac{\omega}{\omega_{zn}})}{(\mathrm{j}\omega)^{N_{p0}}(1+\mathrm{j}\frac{\omega}{\omega_{p1}})(1+\mathrm{j}\frac{\omega}{\omega_{p2}})\cdots(1+\mathrm{j}\frac{\omega}{\omega_{pm}})}| \\
&= 20\log|K| + 20N_{z0}\log|\mathrm{j}\omega| + 20\log|1+\mathrm{j}\frac{\omega}{\omega_{z1}}| + \ldots + 20\log|1+\mathrm{j}\frac{\omega}{\omega_{zn}}| \\
&+ 20N_{p0}\log|\frac{1}{\mathrm{j}\omega}| + 20\log|\frac{1}{1+\mathrm{j}\frac{\omega}{\omega_{p1}}}| + \ldots + 20\log|\frac{1}{1+\mathrm{j}\frac{\omega}{\omega_{pm}}}|
\end{aligned}$$

This is similar with phase, where the overall phase is the sum of each factor's phase (by properties of complex numbers):

$$\begin{aligned}
&\measuredangle K\frac{(\mathrm{j}\omega)^{N_{z0}}(1+\mathrm{j}\frac{\omega}{\omega_{z1}})(1+\mathrm{j}\frac{\omega}{\omega_{z2}})\cdots(1+\mathrm{j}\frac{\omega}{\omega_{zn}})}{(\mathrm{j}\omega)^{N_{p0}}(1+\mathrm{j}\frac{\omega}{\omega_{p1}})(1+\mathrm{j}\frac{\omega}{\omega_{p2}})\cdots(1+\mathrm{j}\frac{\omega}{\omega_{pm}})} \\
&= \measuredangle K + N_{z0}\measuredangle(\mathrm{j}\omega) + \measuredangle(1+\mathrm{j}\frac{\omega}{\omega_{z1}}) + \ldots + \measuredangle(1+\mathrm{j}\frac{\omega}{\omega_{zn}}) + N_{p0}\measuredangle(\frac{1}{\mathrm{j}\omega}) + \measuredangle(\frac{1}{1+\mathrm{j}\frac{\omega}{\omega_{p1}}}) + \ldots + \measuredangle(\frac{1}{1+\mathrm{j}\frac{\omega}{\omega_{pm}}})
\end{aligned}$$

What this tells us is that if we can analyze poles and zeros independently, we can combine them using addition to plot more complex transfer functions!

To start out, we will plot the Bode plot of a single pole:

$$H(\mathrm{j}\omega) = \frac{1}{1 + \mathrm{j}\frac{\omega}{\omega_p}} \tag{3}$$

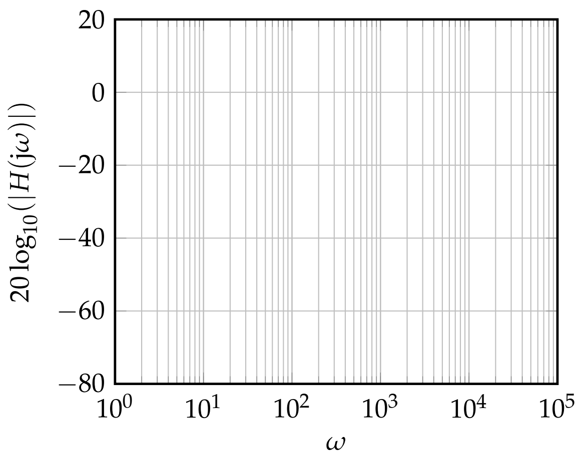
(a) Find and simplify expressions for $|H(\mathrm{j}\omega)|_{\mathrm{dB}}$ and $\measuredangle H(\mathrm{j}\omega)$ as much as possible.

(b) Approximate $|H(\mathrm{j}\omega)|_{\mathrm{dB}}$ for $\omega < \omega_p$ and $\omega > \omega_p$. What is the slope of each section (in units of $\frac{\mathrm{dB}}{\mathrm{dec}}$)?

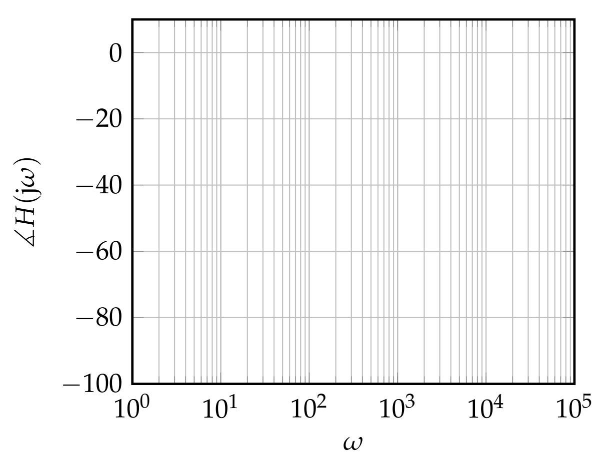
(c) Approximate $\measuredangle H(\mathrm{j}\omega)$ for $\omega < \frac{1}{10}\omega_p$ and $\omega > 10\omega_p$. Calculate the exact value at $\omega = \omega_p$.

(d) Draw the Bode plots for this single pole transfer function on the below plots. You can assume $\omega_p = 10^3 \frac{\text{rad}}{\text{s}}$. *(HINT: Use the same approximation ranges as provided in the previous problems and only draw linear segments. For the phase plot, make the plot linear and pass through the $\omega = \omega_p$ value for the range $\frac{1}{10}\omega_p < \omega < 10\omega_p$.)*

Plot of $|H(\mathrm{j}\omega)|_{dB}$ (for you to draw).



Plot of $\measuredangle H(\mathrm{j}\omega)$ (for you to draw).



From our plots, we should be able to notice that for the magnitude plot, the pole does not contribute (0dB) until $\omega = \omega_p$, at which point the pole cause the slope to decrease by $20\frac{\text{dB}}{\text{dec}}$. For the phase plot, the pole causes the phase to decrease linearly by 90° from $\frac{1}{10}\omega_p$ to $10\omega_p$.

If there is a pole at $\omega_p = 0$, the $-20\frac{\text{dB}}{\text{dec}}$ slope and -90° phase are applied immediately (since you are always past $\omega_p = 0$ in log scale).

Now, we will do the same for a single zero:

$$H(\mathrm{j}\omega) = 1 + \mathrm{j}\frac{\omega}{\omega_z} \tag{4}$$

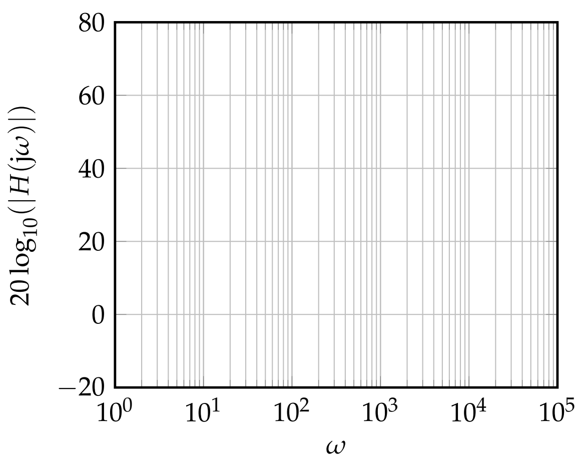
(e) Find and simplify expressions for $|H(\mathrm{j}\omega)|_{\mathrm{dB}}$ and $\measuredangle H(\mathrm{j}\omega)$ as much as possible.

(f) Approximate $|H(\mathrm{j}\omega)|_{\mathrm{dB}}$ for $\omega < \omega_z$ and $\omega > \omega_z$. What is the slope of each section (in units of $\frac{\mathrm{dB}}{\mathrm{dec}}$)?

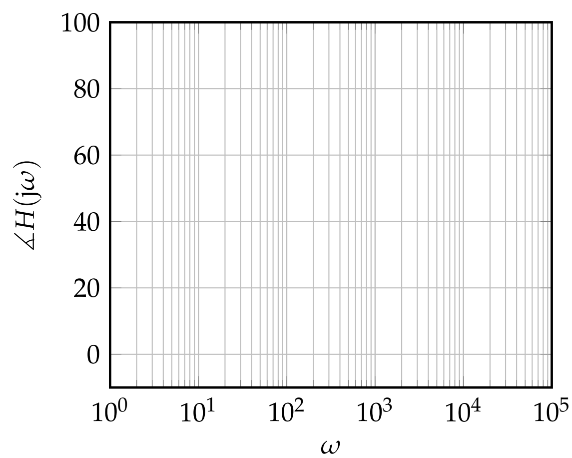
(g) Approximate $\measuredangle H(j\omega)$ for $\omega < \frac{1}{10}\omega_z$ and $\omega > 10\omega_z$. Calculate the exact value at $\omega = \omega_z$.

(h) Draw the Bode plots for this single pole transfer function on the below plots. You can assume $\omega_z = 10^3 \frac{\text{rad}}{\text{s}}$. *(HINT: Use the same approximation ranges as provided in the previous problems and only draw linear segments. For the phase plot, make the plot linear and pass through the $\omega = \omega_z$ value for the range $\frac{1}{10}\omega_z < \omega < 10\omega_z$.)*

Plot of $|H(j\omega)|_{dB}$ (for you to draw).



Plot of $\measuredangle H(j\omega)$ (for you to draw).



From our plots, we should be able to notice that for the magnitude plot, the zero does not contribute (0dB) until $\omega = \omega_z$, at which point the zero cause the slope to increase by $20\frac{\text{dB}}{\text{dec}}$. For the phase plot, the zero causes the phase to increase linearly by $90°$ from $\frac{1}{10}\omega_z$ to $10\omega_z$.

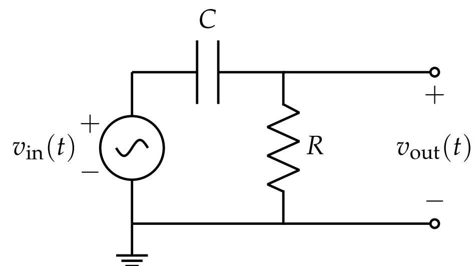
If there is a zero at $\omega_z = 0$, the $+20\frac{\text{dB}}{\text{dec}}$ slope and $+90°$ phase are applied immediately (since you are always past $\omega_z = 0$ in log scale).

2. Bode Plots Practice

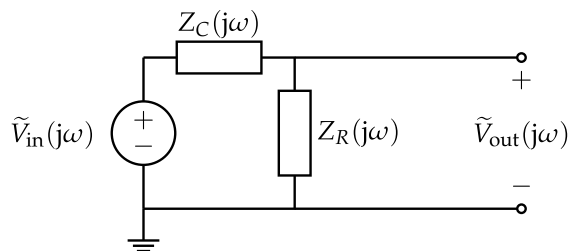
With our knowledge of poles and zeros, let's plot some of the transfer functions we derived in previous discussions!

For all of the following parts, identify the poles and zeros, and plot the Bode plots of the transfer functions on the provided plots.

(a) **RC circuit** ($R = 1\,\text{k}\Omega$, $C = 1\,\mu\text{F}$):

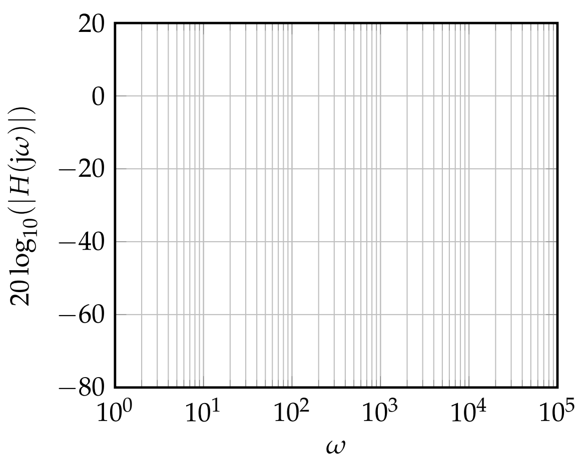


(a) Circuit in "time domain"

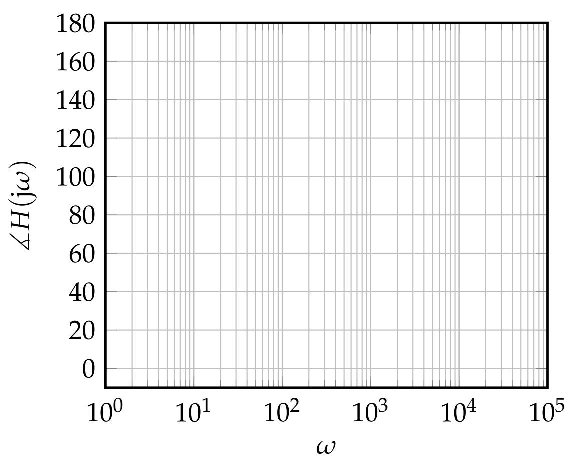


(b) Circuit in "phasor domain"

Plot of $|H(\text{j}\omega)|_{dB}$ (for you to draw).



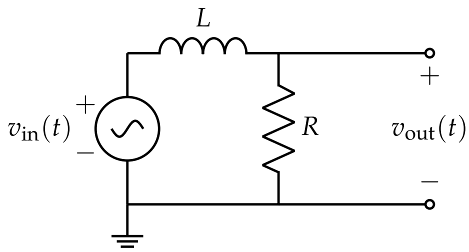
Plot of $\measuredangle H(\text{j}\omega)$ (for you to draw).



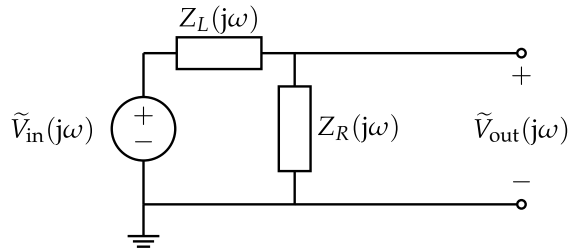
We found that the transfer function for this circuit was:

$$H(\text{j}\omega) = \frac{\text{j}\frac{\omega}{10^3}}{1 + \text{j}\frac{\omega}{10^3}} \tag{5}$$

(b) **LR circuit** ($L = 5\,\text{H}$, $R = 500\,\Omega$):

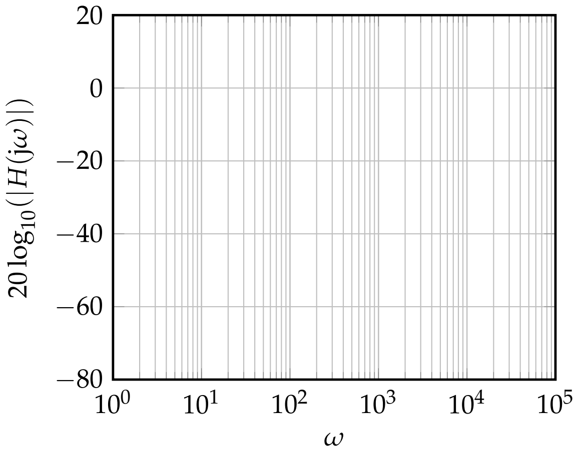


(a) Circuit in "time domain"

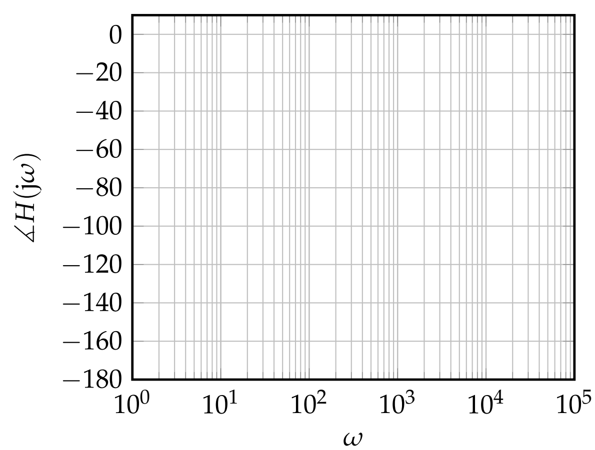


(b) Circuit in "phasor domain"

Plot of $|H(\text{j}\omega)|_{dB}$ (for you to draw).



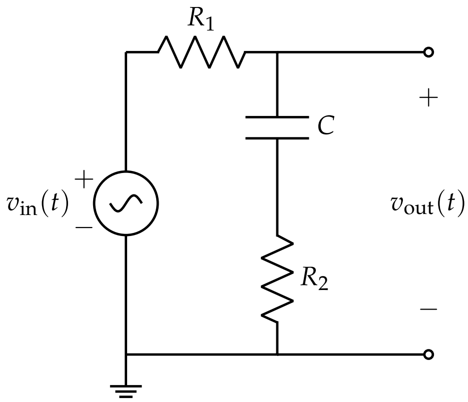
Plot of $\measuredangle H(\text{j}\omega)$ (for you to draw).



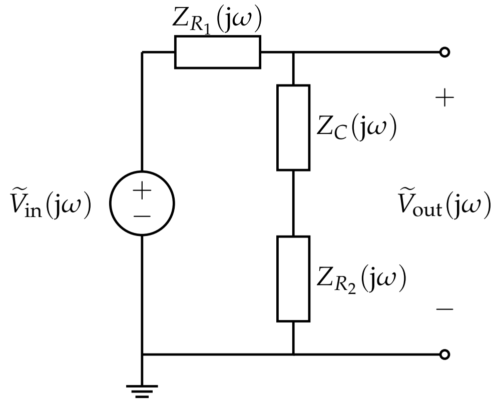
We found that the transfer function for this circuit was:

$$H(\text{j}\omega) = \frac{1}{1 + \text{j}\frac{\omega}{10^2}} \tag{6}$$

(c) **RCR circuit** ($R_1 = 9\,\text{k}\Omega$, $R_2 = 1\,\text{k}\Omega$, $C = 1\,\mu\text{F}$):

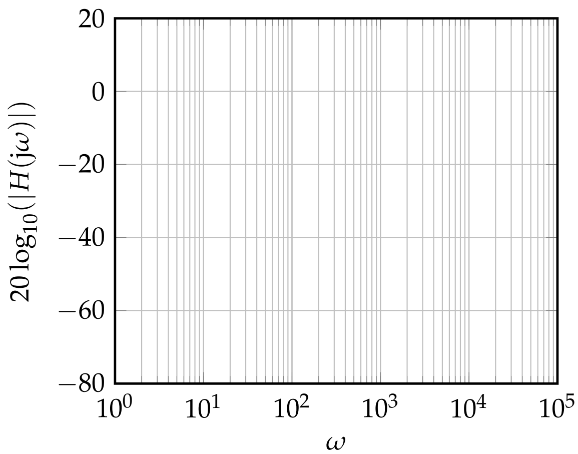


(a) Circuit in "time domain"

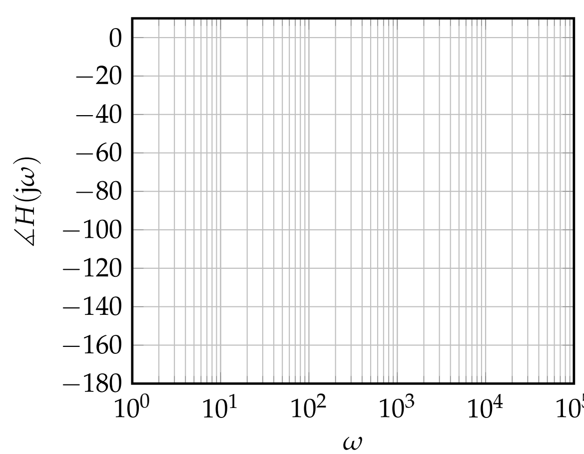


(b) Circuit in "phasor domain"

Plot of $|H(\text{j}\omega)|_{dB}$ (for you to draw).



Plot of $\measuredangle H(\text{j}\omega)$ (for you to draw).



We found that the transfer function for this circuit was:

$$H(\text{j}\omega) = \frac{1 + \text{j}\frac{\omega}{10^3}}{1 + \text{j}\frac{\omega}{10^2}} \tag{7}$$

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