Discussion 4D

The following notes are useful for this discussion: Note 10, Note 11, Note 12

1. Eigenvalue Placement in Discrete Time

Recall that, for a discrete linear system to be stable, we require that all of the eigenvalues of A in $\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i]$ must have magnitude less than 1.

Consider the following linear discrete time system

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i] + \vec{w}[i] \tag{1}$$

(a) **Is the system given in eq. (1) stable?**

Solution: For notation's sake, let's write the system in the familiar form

$$\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i] + \vec{w}[i] \tag{2}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3}$$

We have to calculate the eigenvalues of matrix A. Doing so, we find:

$$\det(A - \lambda I) = 0 \implies \lambda_1 = 1, \lambda_2 = -2 \tag{4}$$

Since there exists a λ such that $|\lambda| \geq 1$ (in fact, both λ_1 and λ_2 satisfy this inequality), the system is unstable.

(b) We can attempt to stabilize the system by implementing closed loop feedback. That is, we choose our input $u[i]$ so that the system is stable. **If we were to use state feedback as in eq. (5), what is an equivalent representation for this system? Write your answer as $\vec{x}[i+1] = A_{\text{CL}}\vec{x}[i]$ for some matrix A_{CL}.**

$$u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] \tag{5}$$

HINT: If you're having trouble parsing the expression for $u[i]$, note that $\begin{bmatrix} f_1 & f_2 \end{bmatrix}$ is a row vector, while $\vec{x}[i]$ is column vector. What happens when we multiply a row vector with a column vector like this?)

Solution: The closed loop system using state feedback has the form

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i] \tag{6}$$

$$= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i] \right) \tag{7}$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} \right) \vec{x}[i] \tag{8}$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \\ 0 & 0 \end{bmatrix} \right) \vec{x}[i] \tag{9}$$

$$= \underbrace{\begin{bmatrix} f_1 & 1+f_2 \\ 2 & -1 \end{bmatrix}}_{A_{\text{CL}}} \vec{x}[i]. \tag{10}$$

(c) **Find the appropriate state feedback constants, f_1, f_2, that place the eigenvalues of the state space representation matrix at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$.**

Solution: From the previous part we have computed the closed loop system as

$$\vec{x}[i+1] = \underbrace{\begin{bmatrix} f_1 & 1+f_2 \\ 2 & -1 \end{bmatrix}}_{A_{\text{CL}}} \vec{x}[i] \tag{11}$$

Thus, finding the eigenvalues of the above system we have

$$0 = \det(A - \lambda I) \tag{12}$$

$$= \det\left(\begin{bmatrix} f_1 - \lambda & 1+f_2 \\ 2 & -1-\lambda \end{bmatrix}\right) \tag{13}$$

$$= \lambda^2 + (1-f_1)\lambda + (-f_1 - 2f_2 - 2) \tag{14}$$

We want to place the eigenvalues at $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = \frac{1}{2}$. This means that we should choose the constants f_1 and f_2 so that the characteristic equation is

$$0 = \left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{1}{2}\right) = \lambda^2 - \frac{1}{4} = \lambda^2 + 0\lambda - \frac{1}{4} \tag{15}$$

Thus, we can match the coefficients of λ in the polynomial above, which indicates we should choose f_1 and f_2 satisfying the following system of equations:

$$0 = 1 - f_1 \tag{16}$$

$$-\frac{1}{4} = -f_1 - 2f_2 - 2 \tag{17}$$

We can solve this two variable, two equation system and find that $f_1 = 1$, $f_2 = -\frac{11}{8}$.

Alternatively, we know what the eigenvalues are; we can plug in each λ into characteristic polynomial, and doing so will yield the same system of equations in f_1, f_2.

(d) **Is the system now stable in closed-loop, using the control feedback coefficients f_1, f_2 that we derived above?**

Solution: Yes, the closed loop system has eigenvalues $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$, which means that A_{CL} satisfies our condition that all of its eigenvalues have magnitude less than 1.

(e) Suppose that instead of $\begin{bmatrix} 1 \\ 0 \end{bmatrix} u[i]$ in eq. (1), we had $\begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i]$ as the way that the discrete-time control acted on the system. In other words, the system is as given in eq. (18). As before, we use $u[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \vec{x}[i]$ to try and control the system.

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i] \tag{18}$$

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Show that the resulting closed-loop state space matrix is

$$A_{\text{CL}} = \begin{bmatrix} f_1 & f_2+1 \\ f_1+2 & f_2-1 \end{bmatrix} \tag{19}$$

Is it possible to stabilize this system?

Solution:

$$\vec{x}[i+1] = \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} f_1 & f_2 \end{bmatrix}\right)\vec{x}[i] \tag{20}$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \\ f_1 & f_2 \end{bmatrix}\right)\vec{x}[i] \tag{21}$$

$$= \underbrace{\begin{bmatrix} f_1 & f_2+1 \\ f_1+2 & f_2-1 \end{bmatrix}}_{A_{\text{CL}}}\vec{x}[i] \tag{22}$$

Finding the eigenvalues λ:

$$0 = \det\left(\begin{bmatrix} f_1-\lambda & f_2+1 \\ f_1+2 & f_2-1-\lambda \end{bmatrix}\right) \tag{23}$$

$$= (f_1-\lambda)(f_2-1-\lambda) - (f_1+2)(f_2+1) \tag{24}$$

$$= f_1(f_2-1) - f_1\lambda - \lambda(f_2-1) + \lambda^2 - (f_1 f_2 + f_1 + 2f_2 + 2) \tag{25}$$

$$= f_1 f_2 - f_1 - f_1\lambda - \lambda f_2 + \lambda + \lambda^2 - f_1 f_2 - f_1 - 2f_2 - 2 \tag{26}$$

$$= \lambda^2 + (1 - f_1 - f_2)\lambda - 2(1 + f_1 + f_2) \tag{27}$$

$$= (\lambda+2)(\lambda - (1 + f_1 + f_2)) \tag{28}$$

We can see that the eigenvalue at $\lambda = -2$ cannot be moved, so we cannot arbitrarily change our eigenvalues with this control input. Since there will always be an eigenvalue with $|\lambda| \geq 1$, then we cannot stabilize the system.

(f) **(PRACTICE)** Suppose you had a discrete, 2D, linear system with a real A matrix, and that you could modify both eigenvalues with feedback control (such as the system in eq. (1)). **Can you place the eigenvalues at complex conjugates, such that $\lambda_1 = a + \mathrm{j}b, \lambda_2 = a - \mathrm{j}b$ using only real feedback gains f_1, f_2? How about placing them at any arbitrary complex numbers, such that $\lambda_1 = a + \mathrm{j}b, \lambda_2 = c + \mathrm{j}d$?**

Solution: Recall that the eigenvalues will be the roots of the polynomial $a\lambda^2 + b\lambda + c$ for some real constants $a, b, c \in \mathbb{R}$. Since all of the coefficients are real, it is not possible to have arbitrary complex eigenvalues — they must either both be real or be complex conjugates. By the quadratic formula, the roots will be $\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, and the only possibly complex quantity can be $\sqrt{b^2-4ac}$. Suppose it is complex. Then, we can write $\sqrt{b^2-4ac} = \mathrm{j}\gamma$ for $\gamma \in \mathbb{R}$. As such, the eigenvalues will be $\lambda_1 = \frac{-b+\mathrm{j}\gamma}{2a}$ and $\lambda_2 = \frac{-b-\mathrm{j}\gamma}{2a}$ which are complex conjugates.

2. Uncontrollability

Recall that, for a n-dimensional, discrete-time linear system to be controllable, we require that the controllability matrix $\mathcal{C} = \begin{bmatrix} A^{n-1}B & A^{n-2}B & \dots & AB & B \end{bmatrix}$ to be rank n.

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Consider the following discrete-time system with the given initial state:

$$\vec{x}[i+1] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i] \tag{29}$$

$$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{30}$$

(a) **Is the system controllable?**

Solution:

$$\mathcal{C} = \begin{bmatrix} A^2B & AB & B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \tag{31}$$

Since the controllability matrix $\mathcal{C}$ only has rank 2, the system is not controllable. We would need it to be rank 3 here to span the full space $\mathbb{R}^3$.

(b) **Show that we can write the ith state as**

$$\vec{x}[i] = \begin{bmatrix} 2^i \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] + 2u[i-1] \end{bmatrix} \tag{32}$$

Is it possible to reach $\vec{x}[\ell] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for some ℓ? If so, for what input sequence $u[i]$ up to $i = \ell - 1$?

Solution: We can write:

$$\vec{x}[i] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[i-1] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i-1] \tag{33}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1[i-1] \\ x_2[i-1] \\ x_3[i-1] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i-1] \tag{34}$$

$$= \begin{bmatrix} 2x_1[i-1] \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] + 2u[i-1] \end{bmatrix} \tag{35}$$

$$= \begin{bmatrix} 2^i x_1[0] \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] + 2u[i-1] \end{bmatrix} \tag{36}$$

$$= \begin{bmatrix} 2^i \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] + 2u[i-1] \end{bmatrix} \tag{37}$$

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Note that in this expression for $\vec{x}[i]$, $x_1[i] = 2^i$ is decoupled from all other states and inputs. From this expression we also see that there is no choice of inputs us to get to $x_1[\ell] = -2$. Therefore, we will never be able to reach $\vec{x}[\ell] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for any ℓ.

(c) **(PRACTICE) Is it possible to reach $\vec{x}[\ell] = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$ for some ℓ? For what input sequence $u[i]$ for $i = 0$ to $i = \ell - 1$?**

HINT: Use the result for $\vec{x}[i]$ from the previous part.

Solution: We need $\ell = 1$ since $x_1[i] = 2^i x_1[0] = 2^i$. Hence,

$$\vec{x}[1] = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} \tag{38}$$

We realize that the first two entries of $\vec{x}[1]$ are exactly what we want. Thus, we have to choose $u[0]$ so the third entry is -2. If we choose $u[0] = -1$, then we have reached our desired state.

Thus we see that a system being uncontrollable does not mean we are unable to reach anything at all, but just that the range that can be reached is limited.

(d) **Find the set of all $\vec{x}[2]$, given that you are free to choose any $u[0]$ and $u[1]$.**

Solution:

$$\vec{x}[1] = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} \tag{39}$$

$$\vec{x}[2] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[1] \tag{40}$$

$$= \begin{bmatrix} 4 \\ -6 + 2u[0] \\ -3 + 2u[1] \end{bmatrix} \tag{41}$$

Since we can set $u[0]$ and $u[1]$ arbitrarily, we can reach any state of the form $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$ after two timesteps. This means that we can reach any value for $\vec{x}[2]$; contrast this with how the first component of the state vector is fixed at 4 after two timesteps, and cannot be changed by the inputs.

Alternative Solution:

Notice that we can write

$$\vec{x}[2] = A^2\vec{x}[0] + ABu[0] + Bu[1] = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u[1] \\ u[0] \end{bmatrix} \tag{42}$$

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Hence, $\vec{x}[2]$ will be $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$ plus whatever is in the column space of

$$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \tag{43}$$

This gives the same answer as before, i.e. that

$$\vec{x}[2] = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \vec{p} \tag{44}$$

where $\vec{p} \in \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$.

Any desired $x_2[i]$ and $x_3[i]$ that we can possibly reach can be obtained in only two or fewer timesteps. Hence, every reachable state can be written as

$$\vec{x}[i] = \begin{bmatrix} 2^i \\ 0 \\ 0 \end{bmatrix} + \vec{p} \tag{45}$$

with $\vec{p}$ defined as above. This will also tell us why the desired goal in part **2.**b is unreachable.

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