

The following notes are useful for this discussion: [Note 10](#) and [Note 11](#).

1. Changing behavior through feedback

In this question, we discuss how *feedback control* can be used to change the effective behavior of a system.

(a) Consider the scalar system:

$$x[i + 1] = 0.9x[i] + u[i] + w[i] \tag{1}$$

where $u[i]$ is the control input we get to apply based on the current state and $w[i]$ is the external disturbance, each at time i .

Is the system stable? If $|w[i]| \leq \epsilon$, what can you say about $|x[i]|$ at all times i if you further assume that $u[i] = 0$ and the initial condition $x[0] = 0$? How big can $|x[i]|$ get?

Solution: The system is stable, as $\lambda = 0.9 \implies |\lambda| < 1$. We can say that $|x[i]|$ is bounded at all time if the disturbance is bounded. Unrolling the system's recursion and extrapolating the general form,

$$x[0] = 0 \tag{2}$$

$$x[1] = w[0] \tag{3}$$

$$x[2] = 0.9w[0] + w[1] \tag{4}$$

$$x[3] = 0.9^2w[0] + 0.9w[1] + w[2] \tag{5}$$

$$\vdots \tag{6}$$

$$x[i] = \sum_{k=0}^{i-1} 0.9^k w[i - k - 1]. \tag{7}$$

We can check that this form works by plugging it into our recursion:

$$x[i + 1] = 0.9x[i] + w[i] = 0.9 \left(\sum_{k=0}^{i-1} 0.9^k w[i - k - 1] \right) + w[i] = \sum_{k=0}^{i-1} 0.9^{k+1} w[i - k - 1] + w[i] = \sum_{k=0}^i 0.9^k w[i - k] \tag{8}$$

which is exactly what our formula predicts. So,

$$|x[i]| = \left| \sum_{k=0}^{i-1} 0.9^k w[i - k - 1] \right| \leq \sum_{k=0}^{i-1} \left| 0.9^k w[i - k - 1] \right| = \sum_{k=0}^{i-1} 0.9^k \epsilon. \tag{9}$$

In the limit as $i \rightarrow \infty$, by the geometric series formula,

$$|x[i]| \leq \frac{\epsilon}{1 - 0.9} = 10\epsilon \tag{10}$$

- (b) Suppose that we decide to choose a control law $u[i] = fx[i]$ to apply in feedback. **Given a specific λ , you want the system to behave like:**

$$x[i + 1] = \lambda x[i] + w[i] \quad (11)$$

To do so, how would you pick f ?

NOTE: In this case, $w[i]$ can be thought of like another input to the system, except we can't control it.

Solution: We can control the system to have any value of λ , as long as we're not limited on the values of f .

$$x[i + 1] = 0.9x[i] + fx[i] + w[i] = \lambda x[i] + w[i]. \quad (12)$$

Fitting terms, $f = \lambda - 0.9$. Note we can get a $\lambda > 1$ if we so desire; there is nothing stopping us from putting arbitrarily big/small λ by the choice of f .

- (c) **For the previous part, which f would you choose to minimize how big $|x[i]|$ can get?**

Solution: From eq. (11), in order to have the minimum bound on $|x[i]|$, $\lambda = 0$. To get this λ , $f = -0.9$. In the limit as $i \rightarrow \infty$ in this case,

$$|x[i]| \leq \frac{\epsilon}{1 - 0} = \epsilon \quad (13)$$

The minimum bound on $|x(i)| = \epsilon$ is the same bound as on the disturbance.

- (d) **What if instead of a 0.9, we had a 3 in the original eq. (1). Would system stability change? Would our ability to control λ change?**

Solution: If our system were now,

$$x[i + 1] = 3x[i] + u[i] + w[i], \quad (14)$$

the system would no longer be stable. However, we can still choose any λ using closed loop feedback. In this case, $f = \lambda - 3$.

- (e) Now suppose that we have a vector-valued system with a vector-valued control:

$$\vec{x}[i + 1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i] \quad (15)$$

where we further assume that B is an invertible square matrix. Further, suppose we decide to apply linear feedback control using a square matrix F so we choose $\vec{u}[i] = F\vec{x}[i]$.

Given a specific A_{CL} we want the system to behave like:

$$\vec{x}[i + 1] = A_{CL}\vec{x}[i] + \vec{w}[i] \quad (16)$$

How would you pick F given knowledge of A, B and the desired goal dynamics A_{CL} ? Will this work for any desired A_{CL} ?

Solution: Since in this case our input is the same rank as our output, we can arbitrarily choose the matrix A_{CL} . As long as B is invertible (as given), we can define:

$$\vec{x}[i + 1] = A\vec{x}[i] + B\vec{u}[i] + \vec{w}[i] \quad (17)$$

$$= A\vec{x}[i] + BF\vec{x}[i] + \vec{w}[i] \quad (18)$$

$$= (A + BF)\vec{x}[i] + \vec{w}[i] \quad (19)$$

$$= A_{\text{CL}}\vec{x}[i] + \vec{w}[i] \quad (20)$$

Therefore, matching terms,

$$A + BF = A_{\text{CL}} \implies F = B^{-1}(A_{\text{CL}} - A). \quad (21)$$

2. Controlling states by designing sequences of inputs

Consider the following matrix, with a simple structure (what does it do when it acts on a vector?):

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (22)$$

Let's assume we have a *discrete-time* system defined as follows:

$$\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]. \quad (23)$$

(a) We are given the initial condition $\vec{x}[0] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Let's say we want to achieve $\vec{x}[\ell] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ for some

specific $\ell \geq 0$ (Note that ℓ denotes total sequence length.) The key is that we want to be in this specific state at this specific timestep, ℓ . **What is the smallest ℓ such that this is possible? What is our choice of sequence of inputs $u[i]$?**

Solution: To ease notation, let

$$\vec{x}[i] = \begin{bmatrix} x_1[i] \\ x_2[i] \\ x_3[i] \\ x_4[i] \end{bmatrix}. \quad (24)$$

Writing out expressions for $x[i]$ we get:

$$\vec{x}[1] = A\vec{x}[0] + \vec{b}u[0] = \begin{bmatrix} x_2[0] \\ x_3[0] \\ x_4[0] \\ u[0] \end{bmatrix}, \quad (25)$$

Note that we could plug in initial conditions here, but we'll leave it general to make the next subparts a bit easier.

$$\vec{x}[2] = \begin{bmatrix} x_3[0] \\ x_4[0] \\ u[0] \\ u[1] \end{bmatrix}, \quad \vec{x}[3] = \begin{bmatrix} x_4[0] \\ u[0] \\ u[1] \\ u[2] \end{bmatrix}, \quad (26)$$

and if $i \geq 4$,

$$\vec{x}[i] = \begin{bmatrix} u[i-4] \\ u[i-3] \\ u[i-2] \\ u[i-1] \end{bmatrix}. \quad (27)$$

Hence, the smallest ℓ is equal to 4 (we applied 4 inputs), with $u[0] = [1]$, $u[1] = [2]$, $u[2] = [3]$, $u[3] = [4]$.

(b) What if we started from $\vec{x}[0] = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$? What is the smallest ℓ and what is our choice of $u[i]$?

Solution: Looking over our expressions for $x[i]$ from the previous part, we see that the earliest ℓ whose expression can be set to the desired state is $\ell = 1$ requiring $u[0] = 4$.

$$\vec{x}[1] = A\vec{x}[0] + \vec{b}u[0] = \begin{bmatrix} x_2[0] \\ x_3[0] \\ x_4[0] \\ u[0] \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ u[0] \end{bmatrix}. \quad (28)$$

(c) If we start from $\vec{x}[0] = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, what is smallest ℓ such that $\vec{x}[\ell] = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, what is corresponding $u[i]$?

Solution: Looking over our expressions for $x[i]$, we see that the earliest ℓ whose expression can be set to the desired state in this case is $\ell = 4$ requiring $u[0] = 1, u[1] = 2, u[2] = 3, u[3] = 4$.

(d) If you would like to make sure that at time ℓ we are at $\vec{x}[\ell] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ for the state, what controls could you use to get there? How big does ℓ have to be for this strategy to work?

Solution: As you might notice, using inputs $u[0] = a, u[1] = b, u[2] = c, u[3] = d$, we can get to

any desired state $\vec{x}[\ell] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. Hence with $\ell = 4$, we can guarantee that the $\vec{x}[\ell]$ is our desired state.

What happens if $\ell > 4$? That is, we know now that we can use 4 timesteps to set the elements of our state-vector however we want, for arbitrary a, b, c, d , but haven't explicitly mentioned what input sequence we can use if, say, $\ell = 8$. Fortunately, we only ever care about the 4 timesteps immediately preceding ℓ , since the effect of anything earlier becomes wiped out by the shifting behavior. Concretely, if $\ell > 4$, we can set all inputs $u[0], u[1], \dots, u[\ell - 5]$ equal to 0; then, $u[\ell - 4] = a, u[\ell - 3] = b, u[\ell - 2] = c, u[\ell - 1] = d$ leads to the desired state at timestep ℓ , which is $x[\ell]$.

Contributors:

- Neelesh Ramachandran.
- Anant Sahai.
- Regina Eckert.
- Kumar Krishna Agrawal.
- Kuan-Yun Lee.
- Kareem Ahmad.