Discussion 4B

1. Transfer Function Practice

Transfer functions take an input phasor and "transform" it into an output phasor. Most of the work we will do with transfer functions is analyzing how it will "respond" to a specific kind of input. We will also design our own transfer functions using common circuit components such as resistors, inductors, and capacitors to achieve some specified behavior. A block diagram of a transfer function is represented below. In this discussion, we will learn how to derive $H(\mathrm{j}\omega)$ from a given circuit, and we will analyze how it behaves for certain values of ω.

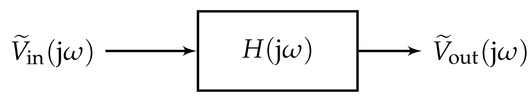


Figure 1: Transfer Function Block Diagram

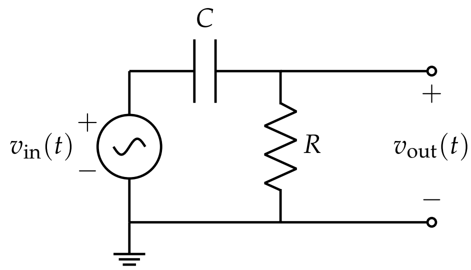
Recall that $Z_L = \mathrm{j}\omega L$ and $Z_C = \frac{1}{\mathrm{j}\omega C}$. For large ω, $|Z_L| = \omega L$ becomes large (and becomes small for small ω). On the other hand, for large ω, $|Z_C| = \frac{1}{\omega C}$ becomes small (and becomes large for small ω).

In this problem, you'll be deriving some transfer functions. For each circuit:

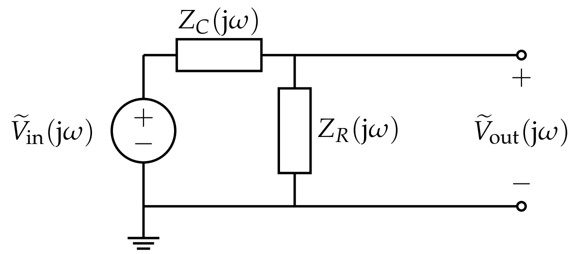
- Determine the **transfer function** $H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}(\mathrm{j}\omega)}{\widetilde{V}_{\text{in}}(\mathrm{j}\omega)}$.
- How does $|H(\mathrm{j}\omega)|$ respond as $\omega \to 0$ **(low frequencies)** and as $\omega \to \infty$ **(high frequencies)**?
- Is the circuit a **high-pass filter, low-pass filter, or band-pass filter**?
- **For parts (a) and (b)**, find the **cutoff frequency** ω_c, which is the frequency such that

$$|H(\mathrm{j}\omega_c)| = \frac{|H(\mathrm{j}\omega)|_{\max}}{\sqrt{2}} \tag{1}$$

(a) **RC circuit** ($R = 1\,\text{k}\Omega$, $C = 1\,\mu\text{F}$):



(a) Circuit in "time domain"



(b) Circuit in "phasor domain"

Solution: We'll use the voltage divider formula to find $\widetilde{V}_{\text{out}}(\text{j}\omega)$:

$$\widetilde{V}_{\text{out}}(\text{j}\omega) = \frac{Z_R}{Z_R + Z_C}\widetilde{V}_{\text{in}}(\text{j}\omega) \tag{2}$$

Recalling the expression for the impendances, we note that for the resistor $Z_R = R$, and for the capacitor $Z_C = \frac{1}{\text{j}\omega C}$. Plugging in the impedances gives

$$H(\text{j}\omega) = \frac{\widetilde{V}_{\text{out}}(\text{j}\omega)}{\widetilde{V}_{\text{in}}(\text{j}\omega)} = \frac{R}{R + \frac{1}{\text{j}\omega C}} = \frac{\text{j}\omega RC}{1 + \text{j}\omega RC} \tag{3}$$

At low frequencies, we have

$$\lim_{\omega \to 0} |H(\text{j}\omega)| = \lim_{\omega \to 0} \frac{\omega RC}{\sqrt{1 + \omega^2 R^2 C^2}} = 0 \tag{4}$$

At high frequencies, we have

$$\lim_{\omega \to \infty} |H(\text{j}\omega)| = \lim_{\omega \to \infty} \frac{\omega RC}{\sqrt{1 + \omega^2 R^2 C^2}} \tag{5}$$

$$= \lim_{\omega \to \infty} \frac{\omega RC}{\sqrt{\omega^2 R^2 C^2}} \tag{6}$$

$$= 1 \tag{7}$$

So this circuit is a high-pass filter.

For this transfer function, $|H(\text{j}\omega)|_{max} = 1$. Thus, to find the cutoff frequency ω_c, we need to find when $|H(\text{j}\omega_c)| = \frac{1}{\sqrt{2}}$.

$$|H(\text{j}\omega_c)| = \frac{1}{\sqrt{2}} \tag{8}$$

$$\frac{\omega RC}{\sqrt{1 + \omega_c^2 R^2 C^2}} = \frac{1}{\sqrt{2}} \tag{9}$$

$$1 + \omega_c^2 R^2 C^2 = 2\omega^2 R^2 C^2 \tag{10}$$

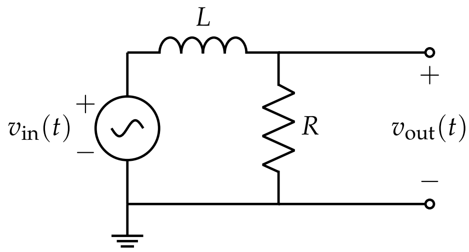
$$\omega_c = \frac{1}{RC} \tag{11}$$

$$= \frac{1}{(10^3)(10^{-6})} = 10^3 \frac{\text{rad}}{\text{s}} \tag{12}$$

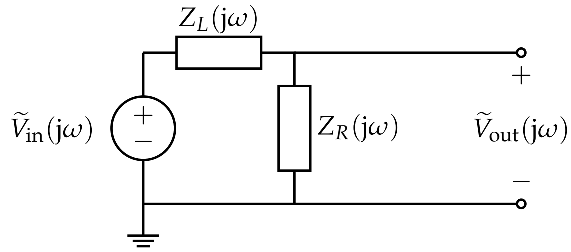
Notice that this can be observed from the transfer function itself by writing it in the following form:

$$\frac{\mathrm{j}\omega RC}{1+\mathrm{j}\omega RC} = \frac{\mathrm{j}\frac{\omega}{\frac{1}{RC}}}{1+\mathrm{j}\frac{\omega}{\frac{1}{RC}}} = \frac{\mathrm{j}\frac{\omega}{\omega_c}}{1+\mathrm{j}\frac{\omega}{\omega_c}} \tag{13}$$

(b) **LR circuit** ($L = 5\,\mathrm{H}$, $R = 500\,\Omega$):



(a) Circuit in "time domain"



(b) Circuit in "phasor domain"

Solution: The strategy is the same as the previous part, using the voltage divider formula, i.e. ,

$$\widetilde{V}_{\text{out}}(\mathrm{j}\omega) = \frac{Z_R}{Z_R + Z_L}\widetilde{V}_{\text{in}}(\mathrm{j}\omega)$$

A similar manipulation to the previous part gives

$$H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}(\mathrm{j}\omega)}{\widetilde{V}_{\text{in}}(\mathrm{j}\omega)} = \frac{R}{R + \mathrm{j}\omega L} \tag{14}$$

At low frequencies, we have

$$\lim_{\omega \to 0} |H(\mathrm{j}\omega)| = \lim_{\omega \to 0} \frac{R}{\sqrt{R^2 + \omega^2 L^2}} = 1 \tag{15}$$

while at high frequencies, we have

$$\lim_{\omega \to \infty} |H(\mathrm{j}\omega)| = \lim_{\omega \to \infty} \frac{R}{R^2 + \omega^2 L^2} = 0 \tag{16}$$

So this circuit is a low-pass filter. Notice that this circuit resembles the one in the previous part, except we have replaced the capacitor with an inductor.

For this transfer function, $|H(\mathrm{j}\omega)|_{max} = 1$. Thus, to find the cutoff frequency ω_c, we need to find when $|H(\mathrm{j}\omega_c)| = \frac{1}{\sqrt{2}}$.

$$|H(\mathrm{j}\omega_c)| = \frac{1}{\sqrt{2}} \tag{17}$$

$$\frac{R}{\sqrt{R^2 + \omega_c^2 L^2}} = \frac{1}{\sqrt{2}} \tag{18}$$

$$R^2 + \omega_c^2 L^2 = 2R^2 \tag{19}$$

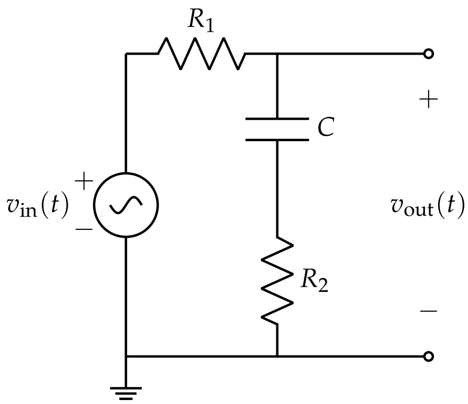
$$\omega_c = \frac{R}{L} \tag{20}$$

$$= \frac{500}{5} = 10^2 \frac{\text{rad}}{\text{s}} \tag{21}$$

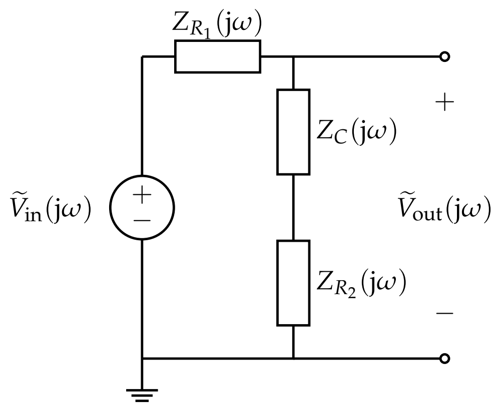
Notice that this can be observed from the transfer function itself by writing it in the following form:

$$\frac{R}{R + \mathrm{j}\omega L} = \frac{1}{1 + \mathrm{j}\frac{\omega}{\frac{R}{L}}} = \frac{1}{1 + \mathrm{j}\frac{\omega}{\omega_c}} \tag{22}$$

(c) **RCR circuit** ($R_1 = 9\,\mathrm{k\Omega}$, $R_2 = 1\,\mathrm{k\Omega}$, $C = 1\,\mathrm{\mu F}$):



(a) Circuit in "time domain"



(b) Circuit in "phasor domain"

Solution: Even though there are three components instead of two, we can still use the voltage divider formula by treating R_2 and C as a single impedance given by $Z = Z_C + Z_{R_2}$, giving us $Z = R_2 + \frac{1}{\mathrm{j}\omega C}$. This would give us

$$\widetilde{V}_{\mathrm{out}}(\mathrm{j}\omega) = \frac{Z}{Z_{R_1} + Z}\widetilde{V}_{\mathrm{in}}(\mathrm{j}\omega) \tag{23}$$

Then, the transfer function is

$$H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\mathrm{out}}(\mathrm{j}\omega)}{\widetilde{V}_{\mathrm{in}}(\mathrm{j}\omega)} = \frac{R_2 + \frac{1}{\mathrm{j}\omega C}}{R_1 + R_2 + \frac{1}{\mathrm{j}\omega C}} = \frac{1 + \mathrm{j}\omega R_2 C}{1 + \mathrm{j}\omega C(R_1 + R_2)} \tag{24}$$

At low frequencies, we have

$$\lim_{\omega \to 0} |H(\mathrm{j}\omega)| = \lim_{\omega \to 0} \frac{\sqrt{1 + (\omega R_2 C)^2}}{\sqrt{1 + (\omega C(R_1 + R_2))^2}} = 1 \tag{25}$$

while at high frequencies, we have

$$\lim_{\omega \to \infty} |H(\mathrm{j}\omega)| = \lim_{\omega \to \infty} \frac{\sqrt{1 + (\omega R_2 C)^2}}{\sqrt{1 + (\omega C(R_1 + R_2))^2}} \tag{26}$$

$$= \lim_{\omega \to \infty} \frac{\sqrt{\frac{1}{\omega^2} + (R_2 C)^2}}{\sqrt{\frac{1}{\omega^2} + (C(R_1 + R_2))^2}} \tag{27}$$

$$= \frac{CR_2}{C(R_1 + R_2)} = \frac{R_2}{R_1 + R_2} \tag{28}$$

So at high frequencies, this circuit behaves like a regular voltage divider with just R_1 and R_2, as if the capacitor had vanished. This circuit is like a combination of a low-pass filter and a voltage divider: low frequency inputs are preserved, and high-frequency signals are diminished.

(d) **Assuming $v_{in}(t) = 12\sin(\omega_{in}t)$ compute the $v_{out}(t)$ using the transfer function computed in part 1.a.** Remember that $R = 1\,\mathrm{k\Omega}$ and $C = 1\,\mu\mathrm{F}$ for this circuit, and assume $\omega_{in} = 1000\,\frac{\mathrm{rad}}{\mathrm{s}}$. In words, what is the effect of the transfer function in part 1.a on the magnitude and phase of the input signal?

Solution: To get $v_{out}(t)$, we must first convert $v_{in}(t)$ into phasor domain to get $\widetilde{V}_{in}(\mathrm{j}\omega)$, then apply the transfer function to get $\widetilde{V}_{out}(\mathrm{j}\omega)$, and then convert back to time domain to get $v_{out}(t)$.

To convert from time domain to phasor domain, we use the definition we derived previously:

$$v_{in}(t) = V_0\cos(\omega t + \theta) \leftrightarrow \widetilde{V}_{in}(\mathrm{j}\omega) = V_0 e^{\mathrm{j}\theta} \tag{29}$$

Firstly, note that $\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$, so we can write $v_{in} = 12\sin(\omega t)$ as $v_{in} = 12\cos\left(\omega t - \frac{\pi}{2}\right)$. Pattern matching with the phasor definition (with $V_0 = 12$ and $\phi = -\frac{\pi}{2}$),

$$\widetilde{V}_{in}(\mathrm{j}\omega) = 12e^{-\mathrm{j}\frac{\pi}{2}} \tag{30}$$

Now, we can find $\widetilde{V}_{out}(\mathrm{j}\omega)$ by multiplying the transfer function with the output phasor. Note that we have to evaluate the transfer function at $\omega = \omega_{in} = 1000\,\frac{\mathrm{rad}}{\mathrm{s}}$ since that is the input angular frequency:

$$H(\mathrm{j}\omega_{in}) = \frac{\mathrm{j}(10^3)(10^3)(10^{-6})}{1 + \mathrm{j}(10^3)(10^3)(10^{-6})} \tag{31}$$

$$= \frac{\mathrm{j}}{1+\mathrm{j}} \tag{32}$$

We will write $H(\mathrm{j}\omega_{in})$ in the form $|H(\mathrm{j}\omega_{in})|e^{\mathrm{j}\angle H(\mathrm{j}\omega_{in})}$, so that multiplying with $\widetilde{V}_{in}(\mathrm{j}\omega)$ will be easier. First, to find $|H(\mathrm{j}\omega_{in})|$:

$$|H(\mathrm{j}\omega_{in})| = \left|\frac{\mathrm{j}}{1+\mathrm{j}}\right| = \frac{1}{\sqrt{2}} \tag{33}$$

Next, to find $\angle H(\mathrm{j}\omega_{in})$:

$$\angle H(\mathrm{j}\omega_{in}) = \angle(\mathrm{j}) - \angle(1+\mathrm{j}) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \tag{34}$$

Hence, $H(\mathrm{j}\omega_{in}) = \frac{1}{\sqrt{2}}e^{\mathrm{j}\frac{\pi}{4}}$, and

$$\widetilde{V}_{out}(\mathrm{j}\omega_{in}) = H(\mathrm{j}\omega_{in})\widetilde{V}_{in}(\mathrm{j}\omega_{in}) = 6\sqrt{2}e^{-\mathrm{j}\frac{\pi}{4}} \tag{35}$$

The last step is changing back to the time domain. For this step, we can use the phasor definition in the reverse direction:

$$v_{out}(t) = 6\sqrt{2}\cos\left(1000t - \frac{\pi}{4}\right) \tag{36}$$

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