1. Using the Transfer Function to Determine the Output (Adapted from Hambley Example 6.1)

The transfer function \( H(j\omega) \) of a filter is shown in Figure 1.

If the input signal is given by

\[
    v_{\text{in}}(t) = 2 \cos(1000t + 40^\circ) + 2 \cos(2000t)
\]  

find an expression for the output of the filter \( v_{\text{out}}(t) \).

**Solution:** Since our input is composed on two sinusoids with different frequencies we need to analyze them separately. Let’s call:

\[
    v_{\text{in},1} = 2 \cos(1000t + 40^\circ) \\
    v_{\text{in},2} = 2 \cos(2000t)
\]

Let’s first analyze the output of \( v_{\text{in},1} \). By inspection, we see that \( \omega = 1000 \). Using the provided graphs of the magnitude and phase, we can determine \( |H(j1000)| = 3 \) and \( \angle H(j1000) = 30^\circ \). Putting this together we have:

\[
    H(j1000) = 3e^{j30^\circ} = \frac{V_{\text{out},1}}{V_{\text{in},1}}
\]

The phasor for the input signal is \( V_{\text{in},1} = 2e^{j40^\circ} \), so solving for the output phasor we have:

\[
    V_{\text{out},1} = H(j1000)V_{\text{in},1} = 3e^{j30^\circ} \times 2e^{j40^\circ}
\]

\[
    = 6e^{j70^\circ}
\]
\[ v_{\text{out},1}(t) = 6 \cos(1000t + 70^\circ) \]  

Converting the output phasor back into a time function, we have:

\[ v_{\text{out},1}(t) = 6e^{j70^\circ} \]  

Now, we will apply the same process for \( v_{\text{in},2} \). We observe that \( \omega = 2000 \). Then using the graphs we know that \( |H(j2000)| = 2 \) and \( \angle H(j2000) = 60^\circ \). We also can represent \( v_{\text{in},2}(t) \) as a phasor which would be \( 2e^{j0^\circ} = 2 \). Putting this together and solving for the output phasor,

\[ V_{\text{out},2} = H(j2000) V_{\text{in},2} \]

\[ = 2e^{j60^\circ} \times 2 \]

\[ = 4e^{j60^\circ} \]

Converting this into a time function, we get:

\[ v_{\text{out},2}(t) = 4 \cos(2000t + 60^\circ) \]

Now combining the two output sinusoids we get:

\[ v_{\text{out}}(t) = 6 \cos(1000t + 70^\circ) + 4 \cos(2000t + 60^\circ) \]
2. RC Filter (Hambley Example 6.3)

Suppose you have the RC circuit shown in Figure 2.

![RC Circuit](image)

Figure 2: RC Lowpass Circuit

(a) **Determine the transfer function** $H(j\omega)$ **of the given circuit.** Then, **classify what type of filter this circuit is.** Recall that the transfer function $H(j\omega)$ is defined as the ratio of the output phasor to the input phasor.

$$H(j\omega) = \frac{V_{\text{out}}}{V_{\text{in}}}$$ \hspace{1cm} (14)

**Solution:** To determine the transfer function, we can apply a sinusoidal input signal with phasor $\tilde{V}_{\text{in}}$ and then analyze the behavior of the circuit as a function of the source frequency $\omega$. The phasor current is given by dividing the input voltage phasor by the complex impedance of the circuit.

$$\tilde{I} = \frac{\tilde{V}_{\text{in}}}{R + \frac{1}{j\omega C}}$$ \hspace{1cm} (15)

The phasor for the output voltage would be the product of the phasor current and the impedance of the capacitor:

$$\tilde{V}_{\text{out}} = \frac{1}{j\omega C} \tilde{I}$$ \hspace{1cm} (16)

Then, substituting in our expression for $\tilde{I}$, we have:

$$\tilde{V}_{\text{out}} = \frac{1}{j\omega C} \times \frac{\tilde{V}_{\text{in}}}{R + \frac{1}{j\omega C}}$$ \hspace{1cm} (17)

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{1}{j\omega C \left( R + \frac{1}{j\omega C} \right)}$$ \hspace{1cm} (18)

$$H(j\omega) = \frac{1}{1 + j\omega RC}$$ \hspace{1cm} (19)

An alternate solution (which can actually be applied for a lot of filters and is more concise than the previous method) involves applying the voltage divider principle with the complex impedances of the circuit.

$$\tilde{V}_{\text{out}} = \frac{Z_C}{Z_R + Z_C} \tilde{V}_{\text{in}}$$ \hspace{1cm} (20)

$$\frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = \frac{1}{R + \frac{1}{j\omega C}}$$ \hspace{1cm} (21)
\[ H(j\omega) = \frac{1}{1 + j\omega RC} \] (22)

Plugging in the values for \( R \) and \( C \), we get that our transfer function is:

\[ H(j\omega) = \frac{1}{1 + j\omega(1000 \times 10 \times 10^{-6})} = \frac{1}{1 + j\frac{\omega}{10^6}} \] (23)

Now, to determine what type of filter this is, we can analyze the behavior for low and high \( \omega \). As \( \omega \to 0 \), notice that the magnitude of the transfer function \( H(j\omega) \) will get closer and closer to 1. On the other hand as \( \omega \to \infty \), we see that the magnitude of the transfer function goes to 0. Therefore, we have the our circuit will attenuate/reduce high frequency and maintain low frequency signals, which leads us to conclude that we have a lowpass RC filter.

(b) Suppose, that you are given:

\[ v_{in}(t) = 5 \cos(10t) + 5 \cos(100t) + 5 \cos(1000t) \] (24)

Find an expression for the output signal \( v_{out}(t) \) (please use approximations to simplify the calculations).

**Solution:** Let’s call \( v_{in,1} = 5 \cos(10t) \), \( v_{in,2} = 5 \cos(100t) \), and \( v_{in,3} = 5 \cos(1000t) \) and their phasors \( \mathbf{V}_{in,1}, \mathbf{V}_{in,2}, \) and \( \mathbf{V}_{in,3} \), respectively.

We have that:

\[ \mathbf{V}_{in,1} = 5e^{j0^\circ} = 5 \] (25)
\[ \mathbf{V}_{in,2} = 5e^{j0^\circ} = 5 \] (26)
\[ \mathbf{V}_{in,3} = 5e^{j0^\circ} = 5 \] (27)

Now, we must analyze the value of the transfer function at each of the different input frequency values. We can notice that \( \omega_1 = 10 \text{ rad} \text{s}^{-1} \), \( \omega_2 = 100 \text{ rad} \text{s}^{-1} \), and \( \omega_3 = 1000 \text{ rad} \text{s}^{-1} \).

\[ H(\omega_1) = H(j10) = \frac{1}{1 + \frac{10}{10^6}} \approx 1 \] (28)
\[ H(\omega_2) = H(j100) = \frac{1}{1 + \frac{100}{10^6}} = \frac{1}{\sqrt{2}} e^{-j45^\circ} \] (29)
\[ H(\omega_3) = H(j1000) = \frac{1}{1 + \frac{1000}{10^6}} \approx \frac{1}{10} e^{-j90^\circ} \] (30)

The output phasor for each input phasor is simply the product of the input phasor with the transfer function evaluated at the given frequency \( \mathbf{V}_{out} = H(j\omega)\mathbf{V}_{in} \). Thus we have:

\[ \mathbf{V}_{out,1} = H(j\omega_1)\mathbf{V}_{in,1} = 1 \times 5 \] (31)
\[ = 5 \] (32)
\[ \mathbf{V}_{out,2} = H(j\omega_2)\mathbf{V}_{in,2} = \frac{1}{\sqrt{2}} e^{-j45^\circ} \times 5 \] (33)
\[ = \frac{5}{\sqrt{2}} e^{-j45^\circ} \] (34)
\[ \mathbf{V}_{out,3} = H(j\omega_3)\mathbf{V}_{in,3} = \frac{1}{10} e^{-j90^\circ} \times 5 \] (35)
\[ = \frac{1}{2} e^{-j90^\circ} \quad (36) \]

Converting the output phasors into their sinusoidal representations, we get:

\[ v_{\text{out},1}(t) = 5 \cos (10t) \quad (37) \]
\[ v_{\text{out},2}(t) = \frac{5}{\sqrt{2}} \cos (100t - 45^\circ) \quad (38) \]
\[ v_{\text{out},3}(t) = \frac{1}{2} \cos (1000t - 90^\circ) \quad (39) \]

Lastly, we sum together the output components to get:

\[ v_{\text{out}} = 5 \cos (10t) + \frac{5}{\sqrt{2}} \cos (100t - 45^\circ) + \frac{1}{2} \cos (1000t - 90^\circ) \quad (40) \]

The important concept to understand here is that each input signal is impacted differently by the filter due to their varying frequencies. You can see that the \( \omega = 10 \ \text{rad/s} \) input signal did not change much in amplitude and phase. On the other hand \( \omega = 100 \ \text{rad/s} \) had its amplitude reduced by a factor of \( \frac{1}{\sqrt{2}} \) and phase shifted by \( -45^\circ \). Lastly the \( \omega = 1000 \ \text{rad/s} \) signal is reduced by a whole order of magnitude \((10^{-1})\), demonstrating the lowpass behavior of the filter.
3. LR Filter (Hambley Exercise 6.5)

Suppose you are given the following circuit:

![LR Circuit Diagram](image)

Figure 3: LR Circuit

Derive the transfer function of this filter, classify what type of filter it is, and determine an expression for the cutoff frequency $\omega_c$. (Note: The cutoff frequency can also be called the -3dB frequency, -3dB point, half-power frequency, break frequency, etc. You should be familiar with these equivalent terms since different books and courses may use different terminology to refer to the same concept.)

Solution: Let’s first convert our circuit into the phasor domain.

Now, we can apply the voltage divider principle to write $V_{\text{out}}$ in terms of $V_{\text{in}}$. You can also apply node-voltage analysis to solve for the transfer function as well.

\[
\begin{align*}
V_{\text{out}} &= \frac{Z_R}{Z_R + Z_L} V_{\text{in}} \\
V_{\text{out}} &= \frac{R}{R + j\omega L} V_{\text{in}} \\
\frac{V_{\text{out}}}{V_{\text{in}}} &= \frac{R}{R + j\omega L} \\
H(j\omega) &= \frac{R}{R + j\omega L} \\
H(j\omega) &= \frac{1}{1 + j\omega L} \\
H(j\omega) &= \frac{1}{1 + j\omega L} \\
\end{align*}
\]

Notice that we can rewrite our transfer function to take on the same form as the low pass filter:

\[
H(j\omega) = \frac{1}{1 + j\omega L} = \frac{1}{1 + j\omega R} \\
\]
Another way to determine that this is a lowpass filter is to analyze the magnitude of the transfer function as $\omega \to 0$ and $\omega \to \infty$. As $\omega \to 0$, we notice that the magnitude of the filter approaches 1, whereas when $\omega \to \infty$, the magnitude of the denominator will go to $\infty$, which indicates that the magnitude of the overall transfer function will go to 0. This behavior of maintaining low frequency signals but reducing high frequency signals makes it a lowpass filter.

For this LR filter, we have that the cutoff frequency is:

$$\omega_c = \frac{R}{L}$$  \hspace{1cm} (47)