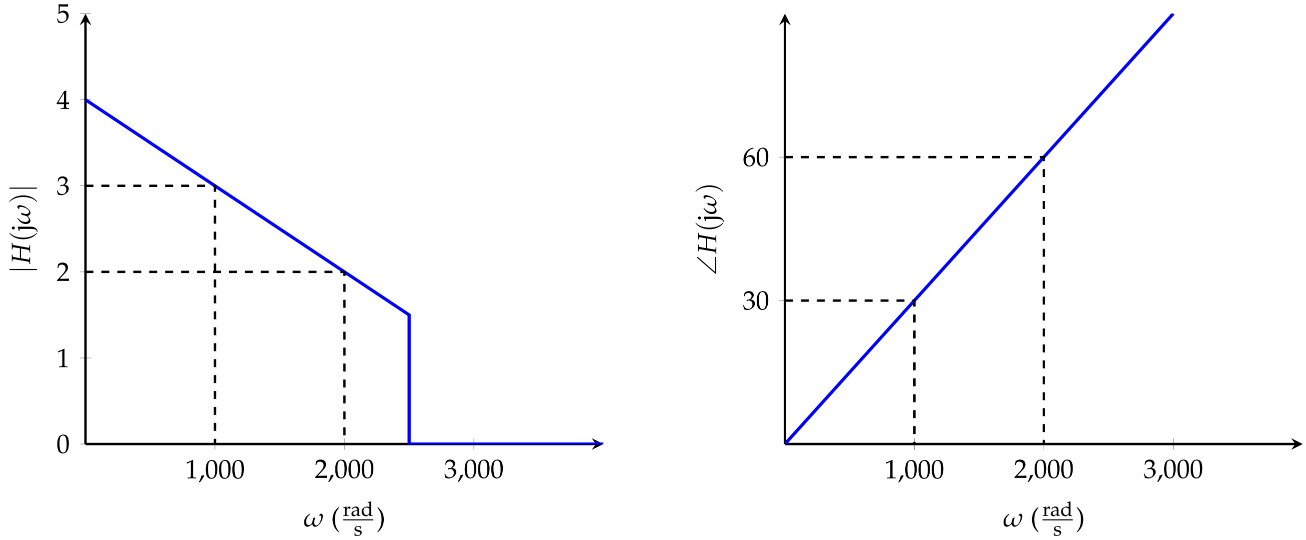
Discussion 4A

1. Using the Transfer Function to Determine the Output (Adapted from Hambley Example 6.1)

The transfer function $H(\mathrm{j}\omega)$ of a filter is shown in Figure 1.



(a) Transfer Function Magnitude

(b) Transfer Function Phase

Figure 1: Transfer Function $H(\mathrm{j}\omega)$

If the input signal is given by

$$v_{\text{in}}(t) = 2\cos(1000t + 40^\circ) + 2\cos(2000t) \tag{1}$$

find an expression for the output of the filter $v_{\text{out}}(t)$.

Solution: Since our input is composed on two sinusoids with different frequencies we need to analyze them separately. Let's call:

$$v_{\text{in},1} = 2\cos(1000t + 40^\circ) \tag{2}$$

$$v_{\text{in},2} = 2\cos(2000t) \tag{3}$$

Let's first analyze the output of $v_{\text{in},1}$. By inspection, we see that $\omega = 1000$. Using the provided graphs of the magnitude and phase, we can determine $|H(\mathrm{j}1000)| = 3$ and $\angle H(\mathrm{j}1000) = 30^\circ$. Putting this together we have:

$$H(\mathrm{j}1000) = 3\mathrm{e}^{\mathrm{j}30^\circ} = \frac{\widetilde{V}_{\text{out},1}}{\widetilde{V}_{\text{in},1}} \tag{4}$$

The phasor for the input signal is $\widetilde{V}_{\text{in},1} = 2\mathrm{e}^{\mathrm{j}40^\circ}$, so solving for the output phasor we have:

$$\widetilde{V}_{\text{out},1} = H(\mathrm{j}1000)\widetilde{V}_{\text{in},1} \tag{5}$$

$$= 3\mathrm{e}^{\mathrm{j}30^\circ} \times 2\mathrm{e}^{\mathrm{j}40^\circ} \tag{6}$$

$$= 6e^{j70^\circ} \tag{7}$$

Converting the output phasor back into a time function, we have:

$$v_{\text{out},1}(t) = 6\cos(1000t + 70^\circ) \tag{8}$$

Now, we will apply the same process for $v_{\text{in},2}$. We observe that $\omega = 2000$. Then using the graphs we know that $|H(\mathrm{j}2000)| = 2$ and $\angle H(\mathrm{j}2000) = 60^\circ$. We also can represent $v_{\text{in},2}(t)$ as a phasor which would be $2e^{j0^\circ} = 2$. Putting this together and solving for the output phasor,

$$\widetilde{V}_{\text{out},2} = H(\mathrm{j}2000)\widetilde{V}_{\text{in},2} \tag{9}$$

$$= 2e^{j60^\circ} \times 2 \tag{10}$$

$$= 4e^{j60^\circ} \tag{11}$$

Converting this into a time function, we get:

$$v_{\text{out},2}(t) = 4\cos(2000t + 60^\circ) \tag{12}$$

Now combining the two output sinusoids we get:

$$v_{\text{out}}(t) = 6\cos(1000t + 70^\circ) + 4\cos(2000t + 60^\circ) \tag{13}$$

2. RC Filter (Hambley Example 6.3)

Suppose you have the RC circuit shown in Figure 2.

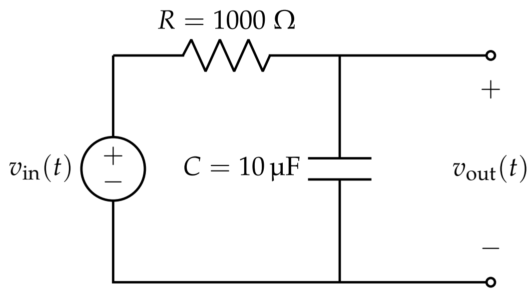


Figure 2: RC Lowpass Circuit

(a) **Determine the transfer function $H(\mathrm{j}\omega)$ of the given circuit. Then, classify what type of filter this circuit is.** Recall that the transfer function $H(\mathrm{j}\omega)$ is defined as the ratio of the output phasor to the input phasor.

$$H(\mathrm{j}\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} \tag{14}$$

Solution: To determine the transfer function, we can apply a sinusoidal input signal with phasor $\widetilde{V}_{\text{in}}$ and then analyze the behavior of the circuit as a function of the source frequency ω. The phasor current is given by dividing the input voltage phasor by the complex impedance of the circuit.

$$\widetilde{I} = \frac{\widetilde{V}_{\text{in}}}{R + \frac{1}{\mathrm{j}\omega C}} \tag{15}$$

The phasor for the output voltage would be the product of the phasor current and the impedance of the capacitor:

$$\widetilde{V}_{\text{out}} = \frac{1}{\mathrm{j}\omega C}\widetilde{I} \tag{16}$$

Then, substituting in our expression for $\widetilde{I}$, we have:

$$\widetilde{V}_{\text{out}} = \frac{1}{\mathrm{j}\omega C} \times \frac{\widetilde{V}_{\text{in}}}{R + \frac{1}{\mathrm{j}\omega C}} \tag{17}$$

$$\frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} = \frac{1}{\mathrm{j}\omega C\left(R + \frac{1}{\mathrm{j}\omega C}\right)} \tag{18}$$

$$H(\mathrm{j}\omega) = \frac{1}{1 + \mathrm{j}\omega RC} \tag{19}$$

An alternate solution (which can actually be applied for a lot of filters and is more concise than the previous method) involves applying the voltage divider principle with the complex impedances of the circuit.

$$\widetilde{V}_{\text{out}} = \frac{Z_C}{Z_R + Z_C}\widetilde{V}_{\text{in}} \tag{20}$$

$$\frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} = \frac{\frac{1}{\mathrm{j}\omega C}}{R + \frac{1}{\mathrm{j}\omega C}} \tag{21}$$

$$H(\mathrm{j}\omega) = \frac{1}{1 + \mathrm{j}\omega RC} \tag{22}$$

Plugging in the values for R and C, we get that our transfer function is:

$$H(\mathrm{j}\omega) = \frac{1}{1 + \mathrm{j}\omega(1000 \times 10 \times 10^{-6})} = \frac{1}{1 + \mathrm{j}\frac{\omega}{100}} \tag{23}$$

Now, to determine what type of filter this is, we can analyze the behavior for low and high ω. As $\omega \to 0$, notice that the magnitude of the transfer function $H(\mathrm{j}\omega)$ will get closer and closer to 1. On the other hand as $\omega \to \infty$, we see that the magnitude of the transfer function goes to 0. Therefore, we have the our circuit will attentuate/reduce high frequency and maintain low frequency signals, which leads us to conclude that we have a lowpass RC filter.

(b) Suppose, that you are given:

$$v_{\text{in}}(t) = 5\cos(10t) + 5\cos(100t) + 5\cos(1000t) \tag{24}$$

Find an expression for the output signal $v_{\text{out}}(t)$ (please use approximations to simplify the calculations).

Solution: Let's call $v_{\text{in},1} = 5\cos(10t)$, $v_{\text{in},2} = 5\cos(100t)$, and $v_{\text{in},3} = 5\cos(1000t)$ and their phasors $\widetilde{V}_{\text{in},1}$, $\widetilde{V}_{\text{in},2}$, and $\widetilde{V}_{\text{in},3}$, respectively.

We have that:

$$\widetilde{V}_{\text{in},1} = 5\mathrm{e}^{\mathrm{j}0^\circ} = 5 \tag{25}$$
$$\widetilde{V}_{\text{in},2} = 5\mathrm{e}^{\mathrm{j}0^\circ} = 5 \tag{26}$$
$$\widetilde{V}_{\text{in},3} = 5\mathrm{e}^{\mathrm{j}0^\circ} = 5 \tag{27}$$

Now, we must analyze the value of the transfer function at each of the different input frequency values. We can notice that $\omega_1 = 10\,\frac{\text{rad}}{\text{s}}$, $\omega_2 = 100\,\frac{\text{rad}}{\text{s}}$, and $\omega_3 = 1000\,\frac{\text{rad}}{\text{s}}$.

$$H(\omega_1) = H(\mathrm{j}10) = \frac{1}{1 + \mathrm{j}\frac{10}{100}} \approx 1 \tag{28}$$
$$H(\omega_2) = H(\mathrm{j}100) = \frac{1}{1 + \mathrm{j}\frac{100}{100}} = \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{j}45^\circ} \tag{29}$$
$$H(\omega_3) = H(\mathrm{j}1000) = \frac{1}{1 + \mathrm{j}\frac{1000}{100}} \approx \frac{1}{10}\mathrm{e}^{-\mathrm{j}90^\circ} \tag{30}$$

The output phasor for each input phasor is simply the product of the input phasor with the transfer function evaluated at the given frequency $\widetilde{V}_{\text{out}} = H(\mathrm{j}\omega)\widetilde{V}_{\text{in}}$. Thus we have:

$$\widetilde{V}_{\text{out},1} = H(\mathrm{j}\omega_1)\widetilde{V}_{\text{in},1} = 1 \times 5 \tag{31}$$
$$= 5 \tag{32}$$
$$\widetilde{V}_{\text{out},2} = H(\mathrm{j}\omega_2)\widetilde{V}_{\text{in},2} = \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{j}45^\circ} \times 5 \tag{33}$$
$$= \frac{5}{\sqrt{2}}\mathrm{e}^{-\mathrm{j}45^\circ} \tag{34}$$
$$\widetilde{V}_{\text{out},3} = H(\mathrm{j}\omega_3)\widetilde{V}_{\text{in},3} = \frac{1}{10}\mathrm{e}^{\mathrm{j}90^\circ} \times 5 \tag{35}$$

$$= \frac{1}{2} e^{-j90^\circ} \tag{36}$$

Converting the output phasors into their sinusoidal representations, we get:

$$v_{\text{out},1}(t) = 5\cos(10t) \tag{37}$$

$$v_{\text{out},2}(t) = \frac{5}{\sqrt{2}}\cos(100t - 45^\circ) \tag{38}$$

$$v_{\text{out},3}(t) = \frac{1}{2}\cos(1000t - 90^\circ) \tag{39}$$

Lastly, we we sum together the output components to get:

$$v_{\text{out}} = 5\cos(10t) + \frac{5}{\sqrt{2}}\cos(100t - 45^\circ) + \frac{1}{2}\cos(1000t - 90^\circ) \tag{40}$$

The important concept to understand here is that each input signal is impacted differently by the filter due to their varying frequencies. You can see that the $\omega = 10\,\frac{\text{rad}}{\text{s}}$ input signal did not change much in amplitude and phase. On the other hand $\omega = 100\,\frac{\text{rad}}{\text{s}}$ had its amplitude reduced by a factor of $\frac{1}{\sqrt{2}}$ and phase shifted by -45°. Lastly the $\omega = 1000\,\frac{\text{rad}}{\text{s}}$ signal is reduced by a whole order of magnitude (10^{-1}), demonstrating the lowpass behavior of the filter.

3. LR Filter (Hambley Exercise 6.5)

Suppose you are given the following circuit:

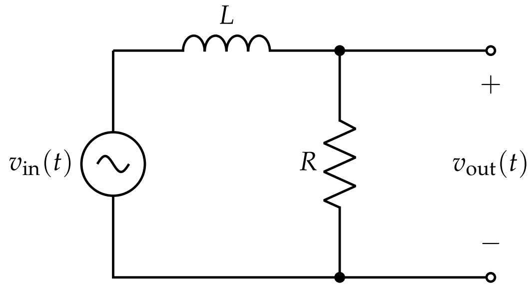


Figure 3: LR Circuit

Derive the transfer function of this filter, classify what type of filter it is, and determine an expression for the cutoff frequency ω_c. (Note: The cutoff frequency can also be called the -3dB frequency, -3dB point, half-power frequency, break frequency, etc. You should be familiar with these equivalent terms since different books and courses may use different terminology to refer to the same concept.)

Solution: Let's first convert our circuit into the phasor domain.

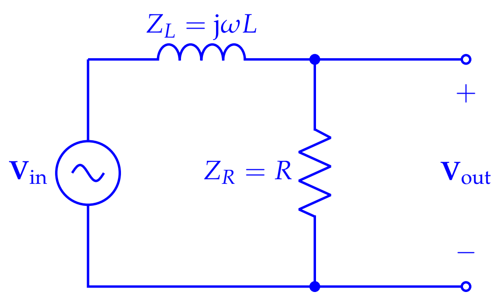


Figure 4: LR Circuit

Now, we can apply the voltage divider principle to write $\widetilde{V}_{\text{out}}$ in terms of $\widetilde{V}_{\text{in}}$. You can also apply node-voltage analysis to solve for the transfer function as well.

$$\widetilde{V}_{\text{out}} = \frac{Z_R}{Z_R + Z_L}\widetilde{V}_{\text{in}} \tag{41}$$

$$\widetilde{V}_{\text{out}} = \frac{R}{R + j\omega L}\widetilde{V}_{\text{in}} \tag{42}$$

$$\frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} = \frac{R}{R + j\omega L} \tag{43}$$

$$H(j\omega) = \frac{R}{R + j\omega L} \tag{44}$$

$$H(j\omega) = \frac{1}{1 + j\omega\frac{L}{R}} \tag{45}$$

Notice that we can rewrite our transfer function to take on the same form as the low pass filter:

$$H(j\omega) = \frac{1}{1 + j\frac{\omega}{\omega_c}} = \frac{1}{1 + j\frac{\omega}{\frac{R}{L}}} \tag{46}$$

Another way to determine that this is a lowpass filter is to analyze the magnitude of the transfer function as $\omega \to 0$ and $\omega \to \infty$. As $\omega \to 0$, we notice that the magntitude of the filter approaches 1, where as when $\omega \to \infty$, the magnitude of the denominator will go to ∞, which indicates that the magnitude of the overall transfer function will go to 0. This behavior of maintaining low frequency signals but reducing high frequency signals makes it a lowpass filter.

For this LR filter, we have that the cutoff frequency is:

$$\omega_c = \frac{R}{L} \tag{47}$$

