Discussion 3B

1. Phasor Analysis Motivation

In this problem, we will try to motivate the reason we use phasors and, in doing so, derive some equations that will useful for phasor analysis.

(a) Suppose we have a capacitor as shown:

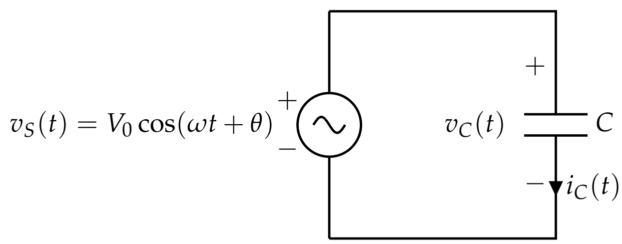

Figure 1

Find the current $i_C(t)$ through the capacitor.

For the rest of the problem, we will use a different method to solve it. Though it may seem more complicated at first, when we have more elements in the circuit, we will see the benefits of understanding this method.

(b) Suppose we have the same circuit, with a different (but related) input (a complex exponential):

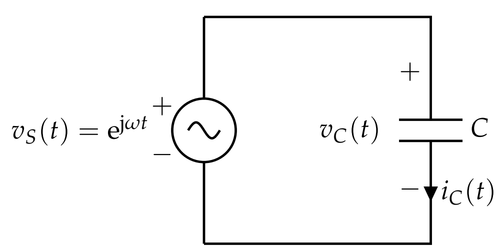

Figure 2

Solve for $i_C(t)$. Then, **calculate the ratio** $Z_C = \frac{v_C(t)}{i_C(t)}$. (Notice that Z_C will depend on the frequency ω, but not on time t. Thus, for a constant ω, Z_C is a constant complex value, similar to resistance. Z_C is called the **impedance** of the capacitor and will be very useful in the future.)

We can show (see previous discussions or notes for more details) that

$$v_S(t) = V_0 \cos(\omega t + \theta) = \frac{V_0}{2}\mathrm{e}^{\mathrm{j}\theta}\mathrm{e}^{\mathrm{j}\omega t} + \frac{V_0}{2}\mathrm{e}^{-\mathrm{j}\theta}\mathrm{e}^{-\mathrm{j}\omega t} \tag{1}$$

Let's define $\widetilde{V}_S = V_0\mathrm{e}^{\mathrm{j}\theta}$. Then, $\overline{\widetilde{V}_S} = V_0\mathrm{e}^{-\mathrm{j}\theta}$ (the complex conjugate of $\widetilde{V}_S$), both of which are constant complex numbers. Using this definition, let's rewrite $v_S(t)$:

$$v_S(t) = \frac{1}{2}\widetilde{V}_S\mathrm{e}^{\mathrm{j}\omega t} + \frac{1}{2}\overline{\widetilde{V}_S}\mathrm{e}^{-\mathrm{j}\omega t} \tag{2}$$

Now, we can use superposition to consider each input separately and combine the outputs.
Let $i_1(t)$ be the current through the capacitor when $v_S(t) = \mathrm{e}^{\mathrm{j}\omega t}$.
Let $i_2(t)$ be the current through the capacitor when $v_S(t) = \mathrm{e}^{-\mathrm{j}\omega t}$.
Then, the total current through the capacitor due to $v_S(t) = \frac{1}{2}\widetilde{V}_S\mathrm{e}^{\mathrm{j}\omega t} + \frac{1}{2}\overline{\widetilde{V}_S}\mathrm{e}^{-\mathrm{j}\omega t}$ is:

$$i_C(t) = \frac{1}{2}\widetilde{V}_S i_1(t) + \frac{1}{2}\overline{\widetilde{V}_S} i_2(t) \tag{3}$$

(You should have found that $Z_C = \frac{1}{\mathrm{j}\omega C}$. This will be used in the following part.)

(c) By evaluating $i_1(t)$ and $i_2(t)$, **show that** $i_C(t) = \frac{1}{2}\left(\frac{\widetilde{V}_S}{Z_C}\mathrm{e}^{\mathrm{j}\omega t} + \frac{\overline{\widetilde{V}_S}}{\overline{Z_C}}\mathrm{e}^{-\mathrm{j}\omega t}\right)$.

(HINT: You already solved for $i_1(t)$ in a previous part. Solving for $i_2(t)$ should be a similar process.)

At this point, we could convert back to a sinusoidal function. However, let's analyze our output.

We will use the following idea: if the input to the system is a sinusoidal function with frequency ω, the currents and voltages throughout the system **in steady state** will also be **sinusoidal functions of the same frequency** ω.

Using this idea, we can say that our output current $i_C(t) = I_0 \cos(\omega t + \phi)$, where ω is the same frequency as in the input $v_S(t) = V_0 \cos(\omega t + \theta)$.

Similar to $v_S(t)$, we can write $i_C(t)$ using complex exponentials, where we define $\widetilde{I}_C = I_0 e^{j\phi}$:

$$i_C(t) = \frac{1}{2}\widetilde{I}_C e^{j\omega t} + \frac{1}{2}\overline{\widetilde{I}_C} e^{-j\omega t} \tag{4}$$

$$= \frac{1}{2}\left(\widetilde{I}_C e^{j\omega t} + \overline{\widetilde{I}_C} e^{-j\omega t}\right) \tag{5}$$

If we compare this expression to the one from part (c), we can identify that $\widetilde{I}_C = \frac{\widetilde{V}_S}{Z_C}$. This looks just like Ohm's Law, where Z_C plays the role of the "resistance" of the capacitor!

Notice that if we know $\widetilde{I}_C$ and the frequency ω, we can construct $i_C(t)$. We just saw that $\widetilde{I}_C = \frac{\widetilde{V}_S}{Z_C}$, and the frequency ω is the same as that of the input $v_S(t)$.

This provides an idea: what if we found $\widetilde{V}_S$, found $\widetilde{I}_C$ using the Ohm's Law relationship, and then converted back to the sinusoidal function represented by $\widetilde{I}_C$ and the frequency ω from our input? This is the idea behind **phasors (or the frequency domain)**, an incredibly useful tool for circuit analysis.

The phasor (or frequency domain) representation of a sinusoidal function is:

$$A\cos(\omega t + \theta) \leftrightarrow A e^{j\theta} \tag{6}$$

The idea is that we will be converting the entire circuit to phasors; sinusoidal sources turn into **constant sources** with the complex valued phasor as defined using the transformation above. All of the resistors, capacitors, and inductors in the circuit behave as "resistors" with a "resistance" equal to their impedance value at the given frequency. Here are the impedance values for each element (Z_R for resistors, Z_C for capacitors, Z_L for inductors):

$$Z_R = R \tag{7}$$

$$Z_C = \frac{1}{j\omega C} \tag{8}$$

$$Z_L = j\omega L \tag{9}$$

Once this is done, the circuit can be analyzed without any differential equations; it is essentially the same as a resistor network with sources! (Remember though that when we do phasor analysis, we are solving for the **steady state** response of the system.)

(d) Let's try this with our example. Once we convert the circuit to phasors, our circuit can be represented like this:

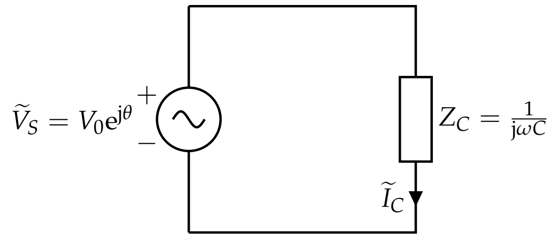

Figure 3

Remember that the capacitor element can now be treated similar to resistor ($\widetilde{V}_C = \widetilde{I}_C Z_C$).

Solve for the phasor $\widetilde{I}_C$. Then, **convert back to the time domain to find** $i_C(t)$.

(HINT: The answer should be the same as in part (a). You can use the facts that $j = e^{j\frac{\pi}{2}}$ *to simplify your phasor and* $\cos\left(\theta + \frac{\pi}{2}\right) = -\sin(\theta)$ *to verify that the solutions are equivalent.)*

While this example was relatively simple (only one capacitor), we can see the benefits of understanding phasor analysis when we have more elements (including additional resistors, capacitors, and inductors).

(e) Suppose now we have an RC circuit with a sinusoidal input:

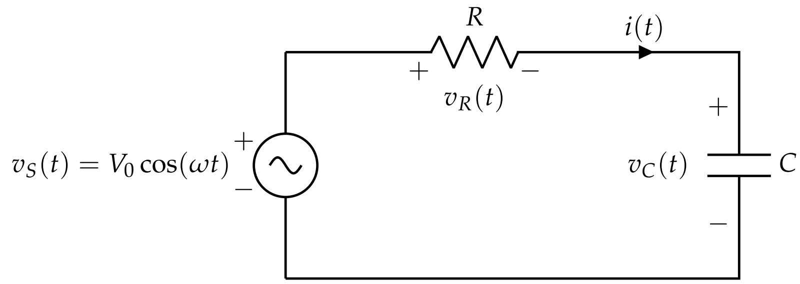

Figure 4

Normally, this would involve a differential equation! However, let's solve this using phasors. Here is the transformed circuit:

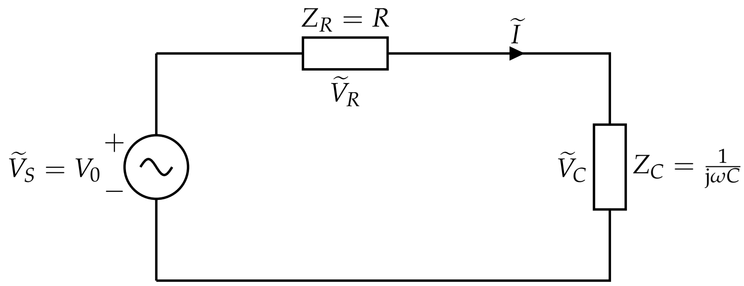

Figure 5

Find the phasor $\widetilde{V}_C$ for the voltage across the capacitor. This will involve no differential equations and we can convert back to the time domain if we want to using some algebra with complex numbers!

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