

## Discussion 2B

## 1. Complex Algebra (Review)

- (a) Express the following values in polar forms:  $-1$ ,  $j$ ,  $-j$ ,  $(j)^{\frac{1}{2}}$ , and  $(-j)^{\frac{1}{2}}$ . Recall  $j^2 = -1$ , and the complex conjugate of a complex number is denoted with a bar over the variable. The complex conjugate is defined as follows: for a complex number  $z = x + jy$ , the complex conjugate  $\bar{z} = x - jy$ .

**Solution:** Here, we review some basic properties of complex numbers and its rectangular and polar form:  $z = x + jy = |z|e^{j\theta}$ , where  $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$  and  $\angle z = \theta = \text{atan2}(y, x)$ . We can also write  $x = |z|\cos(\theta)$ ,  $y = |z|\sin(\theta)$ .

A complex number can be represented in the following forms:

$$z = a + jb = r\cos(\theta) + jr\sin(\theta) = re^{j\theta}, \quad (1)$$

where,  $r = \sqrt{a^2 + b^2}$ ,  $\angle z = \text{atan2}(b, a)$  and  $a, b$  are real numbers.

$$-1 = j^2 = e^{j\pi} = e^{-j\pi} \quad (2)$$

$$j = e^{j\frac{\pi}{2}} = \sqrt{-1} \quad (3)$$

$$-j = -e^{j\frac{\pi}{2}} = e^{-j\frac{\pi}{2}} \quad (4)$$

$$(j)^{\frac{1}{2}} = (e^{j\frac{\pi}{2}})^{\frac{1}{2}} = e^{j\frac{\pi}{4}} = \frac{1+j}{\sqrt{2}} \quad (5)$$

$$(-j)^{\frac{1}{2}} = (e^{-j\frac{\pi}{2}})^{\frac{1}{2}} = e^{-j\frac{\pi}{4}} = \frac{1-j}{\sqrt{2}} \quad (6)$$

- (b) Represent  $\sin(\theta)$  and  $\cos(\theta)$  using complex exponentials. (Hint: Use Euler's identity  $e^{j\theta} = \cos(\theta) + j\sin(\theta)$ .)

**Solution:** Note that we can use the fact that  $\cos(x)$  is an even function, and  $\sin(x)$  is an odd function. This gives us that:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$e^{-j\theta} = \cos(-\theta) + j\sin(-\theta)$$

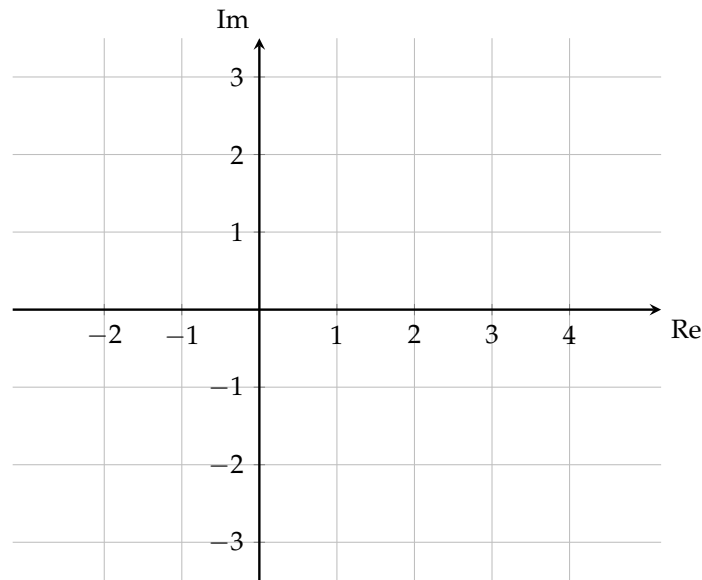
$$= \cos(\theta) - j\sin(\theta)$$

Solving this system of equations for  $\cos(\theta)$  and  $\sin(\theta)$  gives:

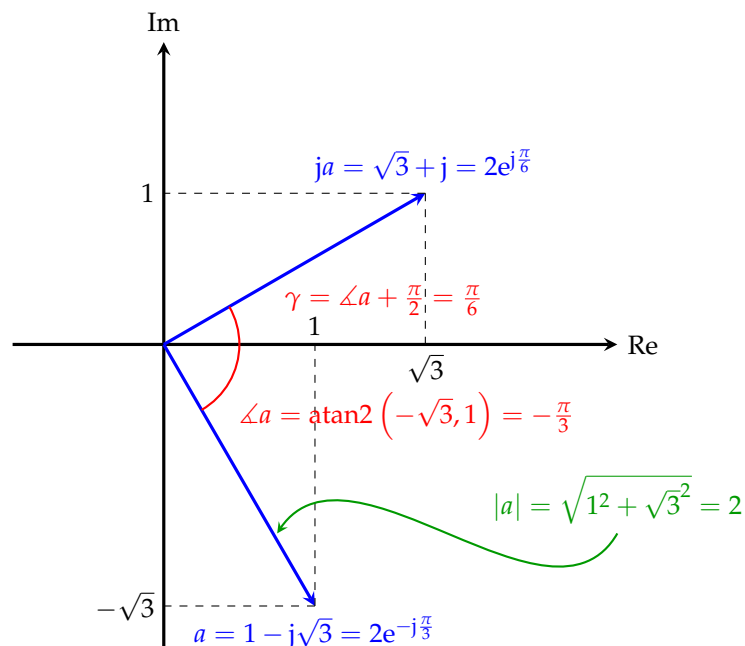
$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \quad \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

For the next parts, let  $a = 1 - j\sqrt{3}$  and  $b = \sqrt{3} + j$ .

- (c) Show the number  $a$  in complex plane, marking the distance from origin and angle with real axis.



**Solution:** The location of  $a$  in the complex plane is shown in Figure 1. The only two pieces of information we need are the magnitude and the phase, which is the polar coordinates interpretation. We could also use the (perhaps more familiar)  $x$  and  $y$  Cartesian coordinates.



**Figure 1:** Complex numbers  $a$  and its rotated version  $b$  represented as vectors in the complex plane.

- (d) Show that multiplying  $a$  with  $j$  is equivalent to rotating the complex number by  $\frac{\pi}{2}$  or  $90^\circ$  in the complex plane.

**Solution:** Multiplying  $a$  by  $j$ :

$$ja = e^{j\pi/2} \cdot 2e^{-j\pi/3} = 2e^{j\pi/6} = \sqrt{3} + j$$

The rotation is demonstrated in the same complex plane plot (Figure 1), with a new angle  $\gamma = \angle a + \frac{\pi}{2}$ .

- (e) For complex number  $z = x + jy$  show that  $|z| = \sqrt{z\bar{z}}$ , where  $\bar{z}$  is the complex conjugate of  $z$ .

**Solution:** We can follow the definition of complex conjugate and magnitude:

$$\sqrt{z\bar{z}} = \sqrt{(x + jy)(x - jy)} = \sqrt{x^2 + y^2} = |z| \quad (7)$$

- (f) Express  $a$  and  $b$  in polar form.

**Solution:** Following the definitions in part a):

$$|a| = 2$$

$$|b| = 2$$

$$\angle a = -\frac{\pi}{3}$$

$$\angle b = \frac{\pi}{6}$$

Hence:

$$a = 2e^{-j\frac{\pi}{3}} \quad b = 2e^{j\frac{\pi}{6}}$$

- (g) Find  $ab$ ,  $a\bar{b}$ ,  $\frac{a}{b}$ ,  $a + \bar{a}$ ,  $a - \bar{a}$ ,  $\overline{ab}$ ,  $\overline{a\bar{b}}$ , and  $(b)^{\frac{1}{2}}$ .

**Solution:** We can evaluate these sequentially using the rules of complex algebra:

$$ab = 4 \cdot e^{-j\frac{\pi}{6}} = 2\sqrt{3} - 2j$$

$$a\bar{b} = 4 \cdot e^{-j\frac{\pi}{2}} = -4j$$

$$\frac{a}{b} = e^{-j\frac{\pi}{2}} = -j$$

$$a + \bar{a} = 2$$

$$a - \bar{a} = -2j\sqrt{3}$$

$$\overline{ab} = 2\sqrt{3} + 2j$$

$$\overline{a\bar{b}} = (1 + j\sqrt{3})(\sqrt{3} - j) = \sqrt{3} + \sqrt{3} + j(3 - 1) = 2\sqrt{3} + 2j$$

$$(b)^{\frac{1}{2}} = \sqrt{2}e^{j\frac{\pi}{12}}$$

Note the following:  $a + \bar{a}$  is a purely real number.  $a - \bar{a}$  is a purely imaginary number. And,  $\overline{a\bar{b}} = \overline{ab}$ .

## 2. Introduction to Inductors

An inductor is a circuit element analogous to a capacitor; its voltage is proportional to the derivative of the current across it. That is:

$$V_L(t) = L \frac{dI_L(t)}{dt} \quad (8)$$

When first studying capacitors, we analyzed a circuit where a current source was directly attached to a capacitor. In Figure 2, we form the counterpart circuit for an inductor:

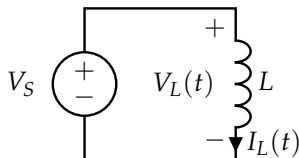


Figure 2: Inductor in series with a voltage source.

- (a) **What is the current through an inductor as a function of time? If the inductance is  $L = 3 \text{ H}$ , what is the current at  $t = 6 \text{ s}$ ?** Assume that the voltage source turns from  $0 \text{ V}$  to  $5 \text{ V}$  at time  $t = 0 \text{ s}$ , and there's no current flowing in the circuit before the voltage source turns on, i.e.  $I_L(0) = 0 \text{ A}$ .

**Solution:** We proceed to analyze the given equation. Note that the voltage source is held at a constant value for  $t \geq 0$ , which allows us to express the derivative of current as a constant:

$$\begin{aligned} V_L(t) &= L \frac{dI_L}{dt} \\ \frac{V_S}{L} &= \frac{dI_L}{dt} \end{aligned}$$

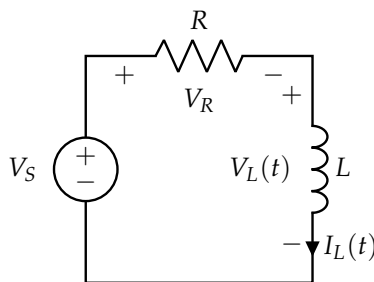
From here, we can see that the derivative of the current is a constant with respect to time! This immediately indicates that we have a linear relationship between current and time, with a slope set by the derivative. This means that the current in the inductor is given by

$$I_L(t) = I_L(0) + \frac{V_S}{L}t \quad (9)$$

This is exactly how we came up with the equation for the voltage across a capacitor in series with a constant current source. So, the current in the inductor keeps growing over time! Inductors store energy in their magnetic field, so the more time that this voltage source feeds the inductor, the higher the current, and the greater the stored energy.

Substituting in the specific values asked for,  $I_L(6 \text{ s}) = \frac{5 \text{ V}}{3 \text{ H}} \cdot 6 \text{ s} = 10 \text{ A}$ .

- (b) Now, we add some resistance in series with the inductor, as in Figure 4.



**Figure 3:** Inductor in series with a voltage source.

**Solve for the current  $I_L(t)$  and voltage  $V_L(t)$  in the circuit over time, in terms of  $R, L, V_S, t$ . Note that  $I_L(0) = 0$  A. Try to solve this equation by inspection. Otherwise, you can use the following integral for the particular solution (with the proper values and functions):**

$$e^{-st} \int e^{st} b(t) dt$$

**Solution:** We begin by considering the voltage drop across the resistor, in terms of source voltage and inductor voltage. There's also only a single current in the circuit (the one we're solving for,  $I(t)$ ):

$$V_R(t) = V_S - V_L(t)$$

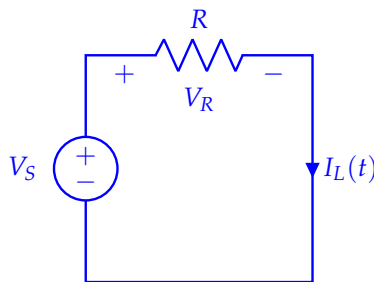
$$R I_L(t) = V_S - L \frac{d}{dt} I_L(t)$$

$$\frac{d}{dt} I_L(t) + \frac{R}{L} I_L(t) = \frac{V_S}{L}$$

We recognize this as a first-order differential equation with an input!

#### Method 1: Inspection

After a long time,  $I_L$  will reach some steady state value. In steady state, an inductor behaves as a short circuit so if we replace the inductor with a short circuit, we can find the steady state current through it. In doing so, we can visualize the following circuit:



**Figure 4:** Inductor in series with a voltage source.

The current in this case would simply be  $\lim_{t \rightarrow \infty} I_L(t) = \frac{V_S}{R}$ .

From our differential equation, we can recognize that our time constant is  $\tau = \frac{L}{R}$ . Additionally, we know that  $I_L$  goes from  $I_L(0) = 0$  to  $\lim_{t \rightarrow \infty} I_L(t) = \frac{V_S}{R}$  exponentially, so the term that describes this transition is  $1 - e^{-\frac{t}{\tau}} = 1 - e^{-\frac{R}{L}t}$ .

Combining our ideas, we can determine that  $i_L(t) = \frac{V_S}{R}(1 - e^{-\frac{R}{L}t})$ .

### Method 2: Homogeneous and Particular Solution

Let's solve the differential equation by finding a homogeneous and particular solution.

Let  $I_h(t)$  be a homogeneous solution. To find it, set the input term to 0 to find the relevant differential equation:

$$\frac{d}{dt}I_h(t) + \frac{R}{L}I_h(t) = 0$$

Notice that this differential equation is similar to the RC differential equation! If we set  $\tau = \frac{L}{R}$ , we can find an identical solution:

$$I_h(t) = A_1 e^{-\frac{t}{\tau}} = A_1 e^{-\frac{R}{L}t}$$

Let  $I_p(t)$  be a particular solution to our differential equation. We can find it either by using the integrating factor or directly using the provided integral (which is derived using the integrating factor). Here, we will directly use the integral (the bounds of the integral will not matter since the constant term from the integral will eventually combine with the arbitrary constant  $A_1$  from the homogeneous solution so you could just evaluate the integral as an indefinite integral, but we will use bounds of 0 and  $t$  for formality).

For our case,  $s = \frac{1}{\tau} = \frac{R}{L}$  and  $b(t) = \frac{V_S}{L}$ .

$$\begin{aligned} I_p(t) &= e^{-\frac{R}{L}t} \int_0^t e^{\frac{R}{L}t'} \frac{V_S}{L} dt' \\ &= e^{-\frac{R}{L}t} \left[ \frac{V_S}{R} e^{\frac{R}{L}t'} \right]_0^t \\ &= e^{-\frac{R}{L}t} \left[ \frac{V_S}{R} e^{\frac{R}{L}t} - \frac{V_S}{R} \right] \\ &= \frac{V_S}{R} - \frac{V_S}{R} e^{-\frac{R}{L}t} \end{aligned}$$

Now, we can combine the two solutions to get the overall solution:

$$\begin{aligned} I_L(t) &= I_h(t) + I_p(t) \\ &= A_1 e^{-\frac{R}{L}t} + \frac{V_S}{R} - \frac{V_S}{R} e^{-\frac{R}{L}t} \\ &= \left( A_1 - \frac{V_S}{R} \right) e^{-\frac{R}{L}t} + \frac{V_S}{R} \\ &= A e^{-\frac{R}{L}t} + \frac{V_S}{R} \end{aligned}$$

Notice that we defined  $A = A_1 - \frac{V_S}{R}$ , which is just another version of the arbitrary constant that will be set by the initial condition (this is why the bounds of the integral were not necessary).

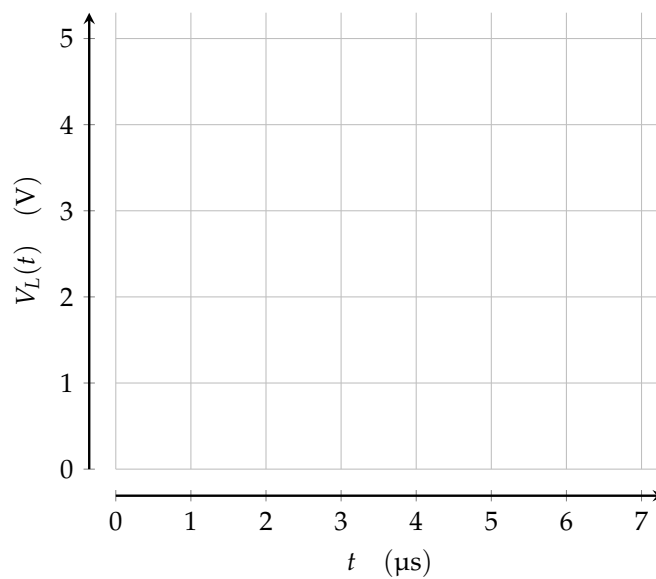
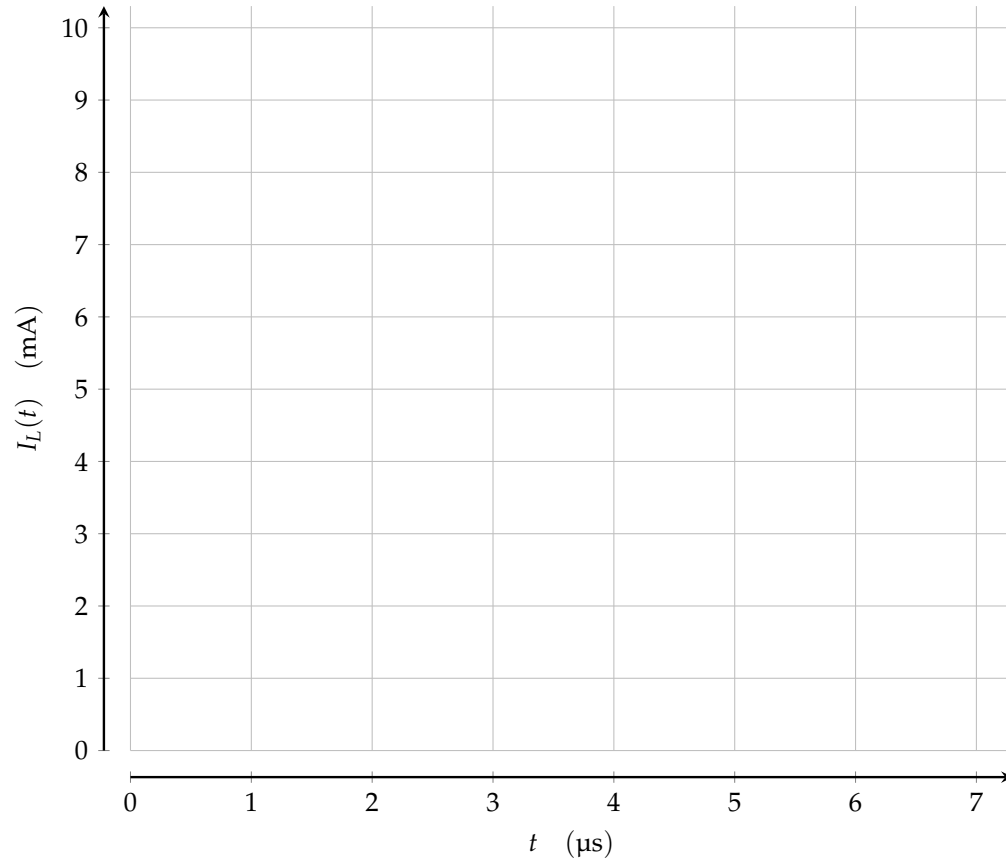
Finally, we can use our initial condition ( $I_L(0) = 0$ ) to solve for  $A$ .

$$\begin{aligned} I_L(0) &= A e^{-\frac{R}{L}(0)} + \frac{V_S}{R} = 0 \\ A &= -\frac{V_S}{R} \end{aligned}$$

Thus, our final solution is

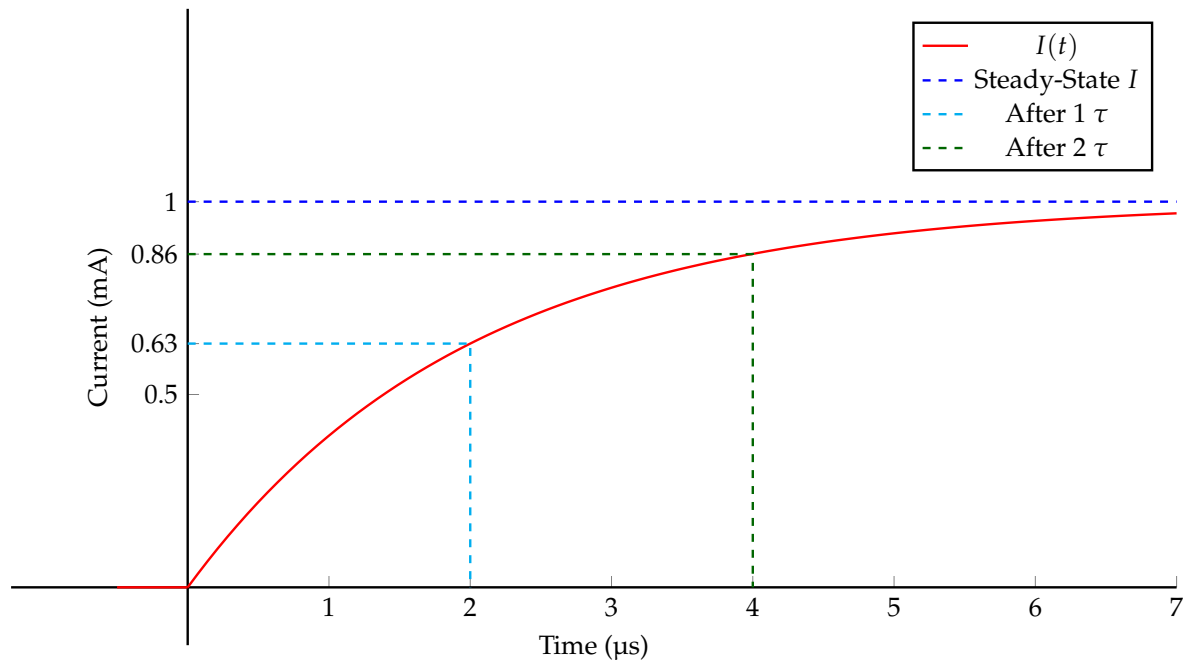
$$I_L(t) = -\frac{V_S}{R}e^{-\frac{R}{L}t} + \frac{V_S}{R} = \frac{V_S}{R} \left(1 - e^{-\frac{R}{L}t}\right)$$

- (c) Suppose  $R = 500\,\Omega$ ,  $L = 1\,\text{mH}$ ,  $V_S = 5\,\text{V}$ . Plot the current through and voltage across the inductor ( $I_L(t)$ ,  $V_L(t)$ ), as these quantities evolve over time.



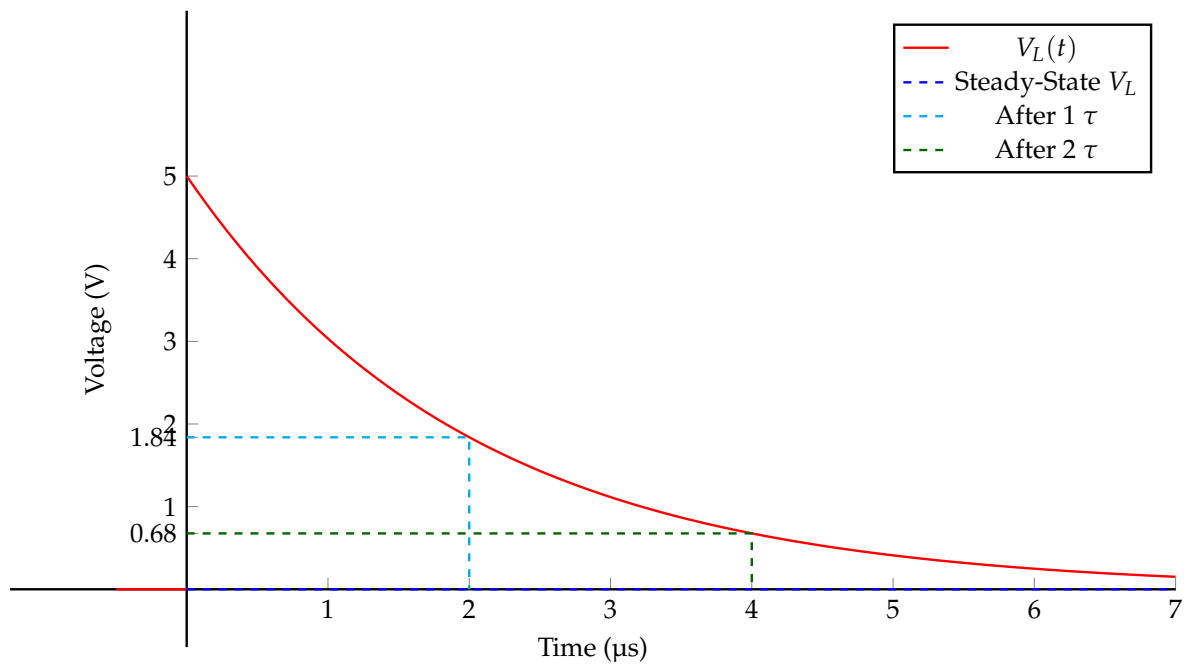
**Solution:** The current begins at 0 A and over time, the inductor begins to look like a short. In the long-term, the current settles to  $\frac{V_S}{R} = 10 \text{ mA}$ . The voltage begins at  $V_S = 5 \text{ V}$  because the inductor initially looks like an open circuit, and this voltage decreases exponentially over time down to zero.

The time constant governing both of these transient curves is  $\tau = \frac{L}{R} = 2 \mu\text{s}$ . Using this information, we can sketch the curves for current (Figure 5) and inductor voltage (Figure 6). Notice that it is perfectly fine for the voltage to be discontinuous, but the same is not true for the current.



**Figure 5:** Transient Current in an RL circuit (with initial current  $I(0) = 0 \text{ A}$ .)





**Figure 6:** Transient Voltage across the inductor in an RL circuit (with initial current  $I(0) = 0 \text{ A}$ .)

**Contributors:**

- Chancharik Mitra.
- Nikhil Jain.
- Neelesh Ramachandran.
- Brian Kilberg.
- Kuan-Yun Lee.
- Maruf Ahmed.
- Anish Muthali.
- Kumar Krishna Agrawal.