Discussion 2B

This discussion relies on material covered in lecture on inductors (**08/31**) and transistors (**09/05**) as well as the corresponding notes, **Note 3** and **Note 4** respectively.

1. RL Circuit Solution Methods

Consider the following circuit:

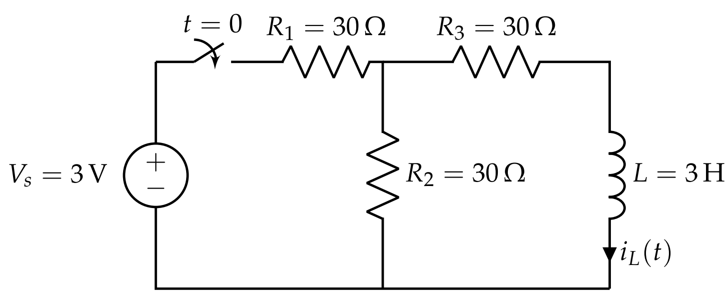


Figure 1

Before time $t = 0$, the circuit reaches a steady state. At time $t = 0$, the switch is closed. Our goal is to find the differential equation for the current through the inductor ($i_L(t)$). One method to approach this problem is to simply use Node Voltage Analysis (NVA). To start, we would define the node voltages in our circuit (including a ground node).

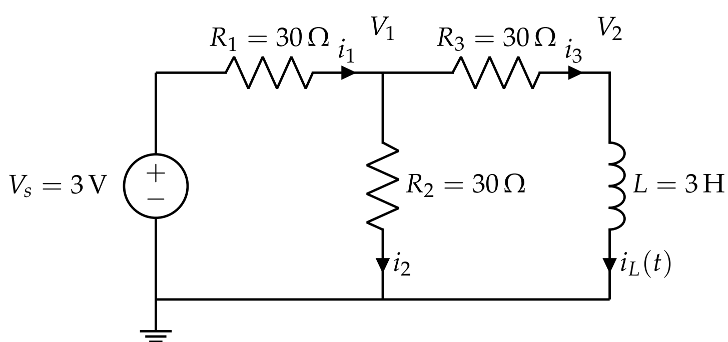


Figure 2

Then, we can set up a system of equations using KCL/KVL to find our desired differential equation.

First, let's perform KCL on the node with defined voltage V_1.

$$
\begin{aligned}
i_1 &= i_2 + i_3 \\
\frac{V_s - V_1}{R_1} &= \frac{V_1 - 0}{R_2} + \frac{V_1 - V_2}{R_3} \\
\frac{3 - V_1}{30} &= \frac{V_1 - 0}{30} + \frac{V_1 - V_2}{30} \\
V_1 &= 1 + \frac{V_2}{3}
\end{aligned}
$$

Now, let's perform KCL on the node with the defined voltage V_2.

Note that $V_2 - 0 = V_2$ is the voltage across the inductor so by the inductor I-V relationship, $V_2 = L\frac{\mathrm{d}i_L}{\mathrm{d}t} = 3\frac{\mathrm{d}i_L}{\mathrm{d}t}$.

$$
\begin{aligned}
i_3 &= i_L \\
\frac{V_1 - V_2}{R_3} &= i_L \\
\frac{V_1 - V_2}{30} &= i_L \\
\frac{V_1}{30} &= \frac{V_2}{30} + i_L \\
\frac{1}{30}\left(1 + \frac{V_2}{3}\right) &= \frac{V_2}{30} + i_L \\
\frac{1}{45}V_2 + i_L &= \frac{1}{30} \\
\frac{1}{45}\left(3\frac{\mathrm{d}i_L}{\mathrm{d}t}\right) + i_L &= \frac{1}{30} \\
\frac{\mathrm{d}i_L}{\mathrm{d}t} + 15i_L &= \frac{1}{2}
\end{aligned}
$$

Thus, we have found the differential equation! However, this method required solving a system of equations; is there another way?

(a) Another way to approach the problem is to use equivalence. Simplify the voltage source and resistor network into a voltage source and resistor using Thevenin equivalence. Then, reconnect the inductor and **find the differential equation for** $i_L(t)$.

For reference, here is the circuit that we want to simplify using Thevenin equivalence:

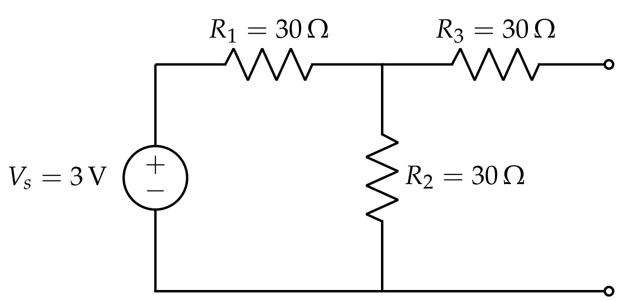


Figure 3

(HINT: Your final differential equation should be the same as the one from the problem introduction.)

Solution: There are many approaches for finding the Thevenin equivalent circuit. Let's find the voltage V_{TH} when the terminals are open and the equivalent resistance R_{TH} looking into the terminals.

To find the voltage V_{TH}, we can notice that no current flows through resistor R_3 due to the open circuit. Thus, the voltage at the terminals is the same as the voltage of the node in between all of the resistors, if we define the bottom node to be ground. Then, since the current through R_1 and R_2 must be the same by KCL, V_{TH} will just be the result of a voltage divider between those two resistors.

$$V_{TH} = \frac{R_2}{R_1 + R_2} V_s = \frac{30}{30 + 30}(3) = \frac{3}{2}$$

To find the equivalent resistance looking into the terminals, we zero out the independent voltage source (which becomes a short circuit) and find the equivalent resistance of the remaining resistors:

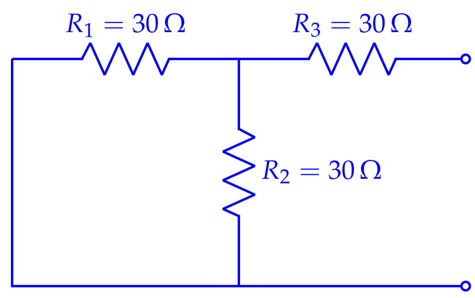


Figure 4

Using parallel/series resistance knowledge, we can find that

$$R_{TH} = R_1 || R_2 + R_3 = 30||30 + 30 = 15 + 30 = 45$$

Thus, our Thevenin equivalent circuit is:

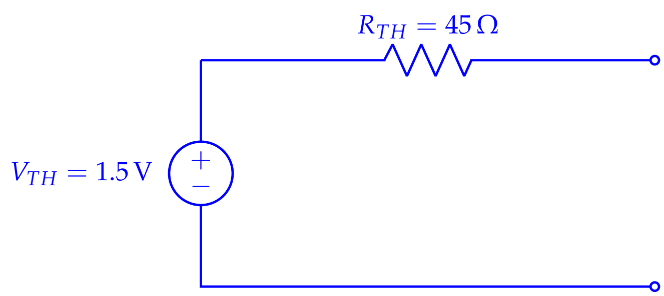


Figure 5

Now, let's add our inductor back into the circuit:

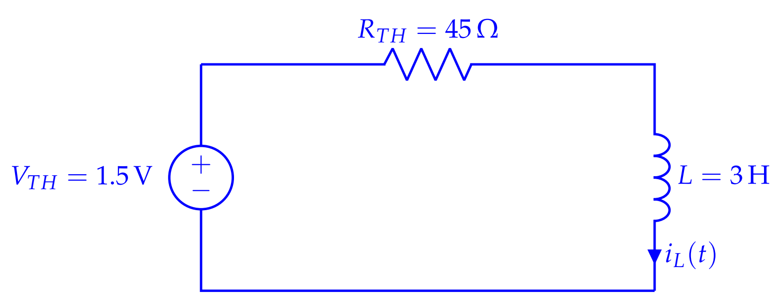


Figure 6

This is a much simpler circuit to analyze! Let's define the voltage across the inductor to be v_L and perform KCL to find the differential equation:

$$\begin{aligned}
\frac{V_{TH} - v_L}{R_{TH}} &= i_L \\
\frac{1.5 - v_L}{45} &= i_L \\
\frac{v_L}{45} + i_L &= \frac{3}{2} \\
\frac{1}{45}\left(3\frac{\mathrm{d}i_L}{\mathrm{d}t}\right) + i_L &= \frac{1}{30} \\
\frac{1}{15}\frac{\mathrm{d}i_L}{\mathrm{d}t} + i_L &= \frac{1}{30} \\
\frac{\mathrm{d}i_L}{\mathrm{d}t} + 15i_L &= \frac{1}{2}
\end{aligned}$$

Notice that this is the same differential equation as obtained using Node Voltage Analysis (NVA)!

(b) Now, let's start solving the differential equation. First, **find the initial condition $i_L(0)$ for our system**. Remember that the current through the inductor cannot change instantaneously (since this would correspond to infinite voltage through the inductor I-V relationship) so $i_L(0)$ will be the same as the steady state value from $t < 0$.

(HINT: If there is no voltage/current sources connected to this system, can there be any nonzero currents / voltage differences in the system during steady-state?)

Solution: Since no voltage/current sources are connected for $t < 0$ when the switch is open, the current in steady state will be $i_L(0) = 0$.

(c) **(OPTIONAL)** Now that we have our differential equation and initial condition, we can now solve for the current $i_L(t)$ as a function of time. **Solve the system for $i_L(t)$**. If you can, try to solve this by inspection. Otherwise, solve using the homogeneous and particular solution method.

Solution:

Method 1: Inspection

We know that when the switch closes, the voltage source becomes connected to the system and after a long time, i_L will reach some steady state value. In steady state, an inductor behaves as a short circuit so if we replace the inductor with a short circuit, we can find the steady state current through it. In doing so, we can visualize the following circuit:

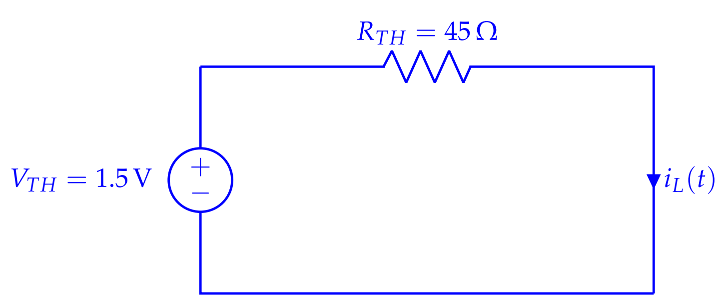


Figure 7

The current in this case would simply be $\lim_{t\to\infty} i_L(t) = \frac{V_{TH}}{R_{TH}} = \frac{1.5}{45} = \frac{1}{30}$.

From our differential equation, we can recognize that our time constant is $\tau = \frac{L}{R_{TH}} = \frac{1}{15}$. Additionally, we know that i_L goes from $i_L(0) = 0$ to $\lim_{t\to\infty} i_L(t) = \frac{1}{30}$ exponentially, so the term that describes this transistion is $1 - e^{-\frac{t}{\tau}} = 1 - e^{-15t}$.

Combining our ideas, we can determine that $i_L(t) = \frac{1}{30}(1 - e^{-15t})$.

Method 2: Homogeneous and Particular Solutions

Notice that our differential equation has an input term (not homogeneous). Thus, we will need to find both a homogeneous solution and particular solution.

Let $i_h(t)$ be a homogeneous solution to our equation. To find $i_h(t)$, set the input term in our differential equation to 0:

$$\frac{\mathrm{d}i_h}{\mathrm{d}t} + 15i_h = 0 \tag{1}$$

$$\frac{\mathrm{d}i_h}{\mathrm{d}t} = -15i_h \tag{2}$$

Notice that this differential equation is the same form as that of RC circuits! If we let $\lambda = -15$, our solution will be identical:

$$i_h(t) = A_1 \mathrm{e}^{\lambda t} = A_1 \mathrm{e}^{-15t}$$

We can also notice that the time constant in this case will be $\tau = \frac{L}{R_{TH}} = \frac{1}{15}$.

Now, let's find a particular solution. For this, we will use the concept of DC steady state (as $t \to \infty$). In DC steady state, an inductor behaves as a short circuit (please refer to the notes and lectures for explanations as to why), so the simplified circuit from out previous example will look as follows in DC steady state:

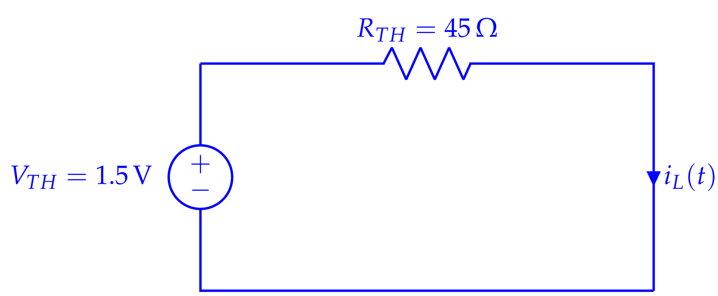


Figure 8

With use of Ohm's law, we can determine that the current through the inductor (represented by the short circuit in this DC steady state scenario) will be:

$$i_p(t) = \frac{1.5}{45} = \frac{1}{30} \tag{3}$$

Now, we can combine the two solutions to get our overall solution.

$$\begin{aligned} i_L(t) &= i_h(t) + i_p(t) \\ &= A_1 \mathrm{e}^{-15t} + \frac{1}{30} - \frac{1}{30}\mathrm{e}^{-15t} \end{aligned}$$

$$\begin{aligned} &= \left(A_1 - \frac{1}{30}\right) \mathrm{e}^{-15t} + \frac{1}{30} \\ &= A\mathrm{e}^{-15t} + \frac{1}{30} \end{aligned}$$

We have defined $A = A_1 - \frac{1}{30}$, which is simply another version of the same arbitrary constant that accounts for the initial condition of our differential equation that we found in the previous part of the problem.

Now, we use our initial condition to solve for A.

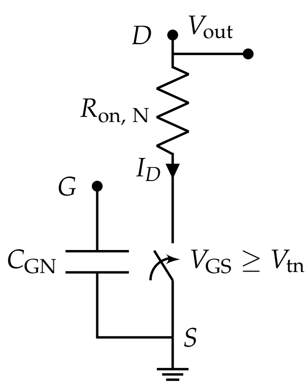
$$\begin{aligned} i_L(0) = A\mathrm{e}^{-15(0)} + \frac{1}{30} &= 0 \\ A &= -\frac{1}{30} \end{aligned}$$

Thus, our final solution is

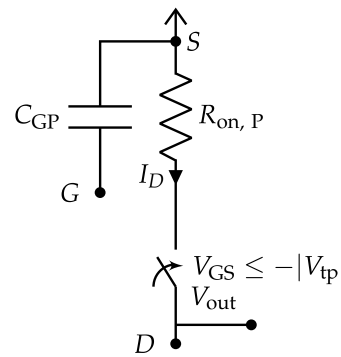
$$i_L(t) = -\frac{1}{30}\mathrm{e}^{-15t} + \frac{1}{30} = \frac{1}{30}\left(1 - \mathrm{e}^{-15t}\right)$$

2. Transistor Switch Model

We can improve our resistor-switch model of the transistor by adding in a gate capacitance. In this model, the gate capacitances C_{GN} and C_{GP} represent the lumped physical capacitance present on the gate node of all transistor devices. This capacitance is important as it determines the delay of a transistor logic chain.



(a) NMOS Transistor Resistor-switch-capacitor model



(b) PMOS Transistor Resistor-switch-capacitor model. Note we have drawn this so that it aligns with the inverter.

You have two CMOS inverters made from NMOS and PMOS devices. Both NMOS and PMOS devices have an "on resistance" of $R_{\text{on, N}} = R_{\text{on, P}} = 1\,\text{k}\Omega$, and each has a gate capacitance (input capacitance) of $C_{\text{GN}} = C_{\text{GP}} = 1\,\text{fF}$ (fF = femto-Farads = 1×10^{-15} F). We assume the "off resistance" (the resistance when the transistor is off) is infinite (i.e., the transistor acts as an open circuit when off). The supply voltage V_{DD} is 1V. Assume $V_{\text{DD}} > V_{\text{tn}}, |V_{\text{tp}}| > 0$. The two inverters are connected in series, with the output of the first inverter driving the input of the second inverter (Figure 10).

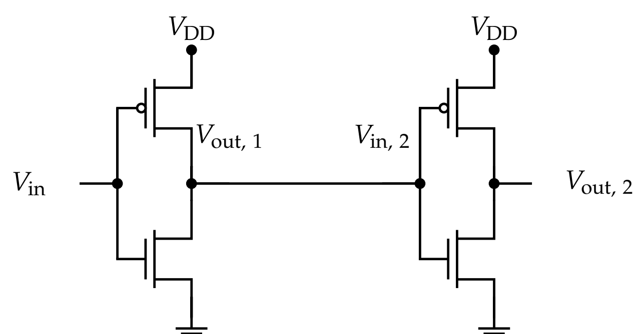


Figure 10: CMOS Inverter chain

(a) Assume the input to the first inverter has been low ($V_{\text{in}} = 0$ V) for a long time, and then switches at time $t = 0$ to high ($V_{\text{in}} = V_{\text{DD}}$).

Draw a simple RC circuit and write a differential equation describing the output voltage of the first inverter ($V_{\text{out, 1}}$) for time $t \geq 0$.

Don't forget that the second inverter is "loading" the output of the first inverter — you need to think about both of them.

(HINT: Your simple RC circuit model will only have 3 elements; you only need to draw the elements that impact the behavior of $V_{\text{out, 1}}$ and thus are relevant in this specific scenario. Also, for the first inverter,

when $V_{\text{in}} = V_{\text{DD}}$, the NMOS transistor model's switch will be closed while the PMOS transistor model's switch will be open.)

Solution: To analyze this circuit as an RC circuit we can recall the transistor switch model. Using this we can see that the first inverter's output appears as a resistor connected to ground when the input turns high ($V_{\text{in}} = V_{\text{DD}}$) since only the switch for the NMOS transistor is closed.

The second inverter "loads" the output of the first inverter. From the notes in the problem, we can model the gates of the transistors as capacitors. These gates together form our capacitive load. The gate of the PMOS acts as a capacitor to V_{DD} and the gate of the NMOS acts as a capacitor to ground.

Using this we can draw the following RC circuit:

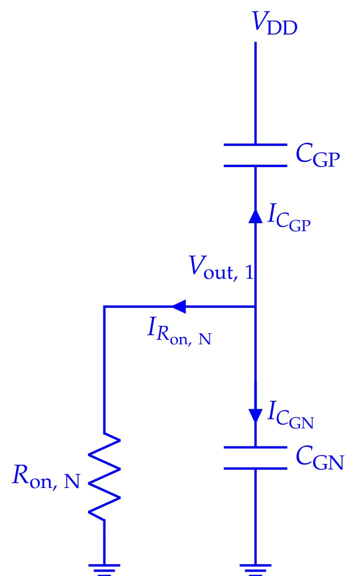


Figure 11: First inverter output at 0

We know the voltage across C_{GP} is $V_{\text{out, 1}}(t) - V_{\text{DD}}$ and the voltage across C_{GN} is $V_{\text{out, 1}}(t)$. Using this information we can set up a differential equation to solve for $V_{\text{out}}(t)$.

Writing the expressions for the three branch currents yields:

$$I_{C_{\text{GP}}} = C_{\text{GP}} \frac{\mathrm{d}}{\mathrm{d}t}(V_{\text{out, 1}}(t) - V_{\text{DD}}) \tag{4}$$

$$I_{C_{\text{GN}}} = C_{\text{GN}} \frac{\mathrm{d}}{\mathrm{d}t} V_{\text{out, 1}}(t) \tag{5}$$

$$I_{R_{\text{on, N}}} = \frac{V_{\text{out, 1}}(t)}{R_{\text{on, N}}} \tag{6}$$

Writing KCL at the single node yields:

$$I_{C_{\text{GP}}} + I_{C_{\text{GN}}} + I_{R_{\text{on, N}}} = 0 \tag{7}$$

in other words:

$$I_{C_{\text{GP}}} + I_{C_{\text{GN}}} = -I_{R_{\text{on, N}}} \tag{8}$$

Expanding the branch currents with their expressions:

$$C_{\text{GP}}\frac{\mathrm{d}}{\mathrm{d}t}(V_{\text{out, 1}}(t) - V_{\text{DD}}) + C_{\text{GN}}\frac{\mathrm{d}}{\mathrm{d}t}V_{\text{out, 1}}(t) = -\frac{V_{\text{out, 1}}(t)}{R_{\text{on, N}}} \tag{9}$$

$$C_{\text{GP}}\frac{\mathrm{d}}{\mathrm{d}t}V_{\text{out, 1}}(t) + C_{\text{GN}}\frac{\mathrm{d}}{\mathrm{d}t}V_{\text{out, 1}}(t) = -\frac{V_{\text{out, 1}}(t)}{R_{\text{on, N}}} \tag{10}$$

$$(C_{\text{GP}} + C_{\text{GN}})\frac{\mathrm{d}}{\mathrm{d}t}V_{\text{out, 1}}(t) = -\frac{V_{\text{out, 1}}(t)}{R_{\text{on, N}}} \tag{11}$$

Re-writing as a first-order differential equation for $V_{\text{out, 1}}$ yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}V_{\text{out, 1}}(t) + \frac{V_{\text{out, 1}}(t)}{R_{\text{on, N}}(C_{\text{GP}} + C_{\text{GN}})} = 0 \tag{12}$$

(b) **Solve for $V_{\text{out, 1}}(t)$.** The initial condition will be $V_{\text{out,1}}(0) = V_{DD}$ (this can be found by using the situation described in part (a)).

Solution: From our differential equation, we can notice that it is in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}V_{\text{out, 1}}(t) + \frac{1}{\tau}V_{\text{out, 1}}(t) = 0 \tag{13}$$

where $\tau = R_{\text{on, N}}(C_{\text{GP}} + C_{\text{GN}})$.

From lecture, you may recognize that this equation is essentially the same as the differential equation for an RC circuit without inputs and the corresponding solution is simply the homogeneous solution for this differential equation

$$V_{\text{out, 1}}(t) = Ke^{-\frac{t}{\tau}} = Ke^{-\frac{t}{R_{\text{on, N}}(C_{\text{GP}}+C_{\text{GN}})}} \tag{14}$$

We can then use our initial condition:

$$V_{\text{out, 1}}(0) = Ke^{-\frac{0}{R_{\text{on, N}}(C_{\text{GP}}+C_{\text{GN}})}} \tag{15}$$

$$V_{\text{DD}} = K \tag{16}$$

Thus, our final solution is

$$V_{\text{out, 1}}(t) = V_{\text{DD}}e^{-\frac{t}{R_{\text{on, N}}(C_{\text{GP}}+C_{\text{GN}})}} \tag{17}$$

(c) **Sketch the output voltage of the first inverter, showing clearly (1) the initial value, (2) the asymptotic value, and (3) the time that it takes for the voltage to decay to roughly 1/3 of its initial value.** *(HINT: For part (3), use the approximation that $\mathrm{e}^{-1} = \frac{1}{\mathrm{e}} \approx \frac{1}{3}$.)*

Solution:

(1) We know that the output of our inverter started with the initial value V_{DD}.

(2) We can find the asymptotic value by plugging in $t = \infty$ to the solution we found for $V_{\text{out, 1}}(t)$ to find $V_{\text{out, 1}} = V_{\text{DD}}e^{-\frac{\infty}{R_{\text{on, N}}(C_{\text{GP}}+C_{\text{GN}})}} = 0$.

(3) To approximate when the output will decay to $\frac{1}{3}$ its original value, we use the fact that $e^{-1} = \frac{1}{e} \approx \frac{1}{3}$. We thus want to find when $V_{\text{out, 1}} = V_{\text{DD}}e^{-1}$.
This will occur when the e term is raised to -1, which occurs when $t = \tau = R_{\text{on, N}}(C_{\text{GP}} + C_{\text{GN}}) = 2 \times 10^{-12}$seconds.
Note the significance of the time constant τ; as defined in the differential equation and solution to the differential equation, it provides a measure of how much time it takes for a system to reach its steady state, which can be compared between different systems to compare their speeds.

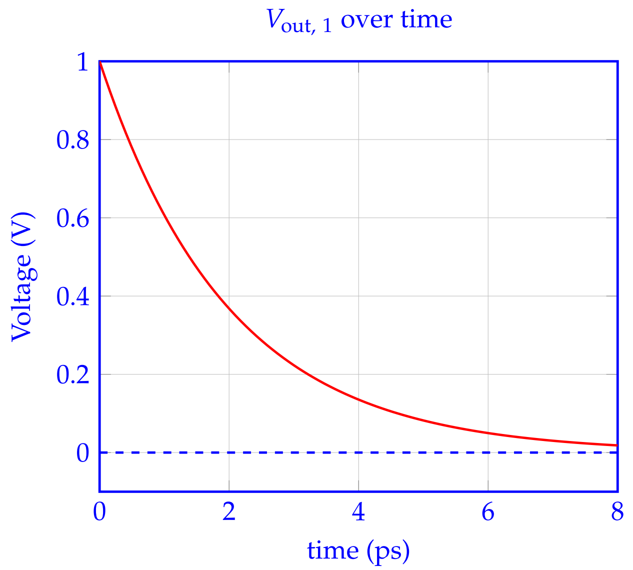


Figure 12

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