Discussion 1D

For this discussion, Note 1 is helpful for the differential equations, and Note j covers the complex numbers fundamentals.

1. **RC Circuits: Solving the Differential Equations**

Recall that in the last discussion, we were tasked with analyzing an example RC circuit (in 1) and using element equations (governing equations for resistors and capacitors) to formulate a differential equation. This equation describes the time-varying behavior of this circuit. Specifically, we had the following differential equation:

$$RC\frac{\mathrm{d}V_C(t)}{\mathrm{d}t} + V_C(t) = V(t) \tag{1}$$

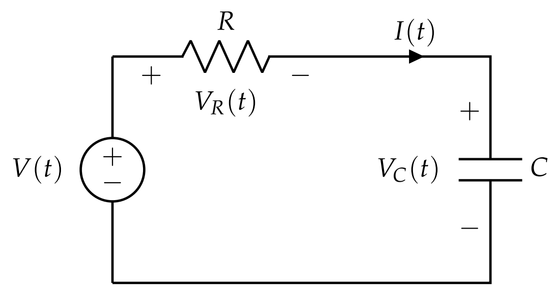


Figure 1: Sample RC Circuit

Our goal is to now solve this differential equation for the voltage across the capacitor, $V_C(t)$. Recall that, in the previous discussion and lecture, we covered two kinds of differential equations:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) \tag{2}$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \lambda x(t) + u \tag{3}$$

where $\lambda, u \in \mathbb{R}$. Eq. (2) has a solution of the form

$$x(t) = A\mathrm{e}^{bt} \tag{4}$$

for some constants $A, b \in \mathbb{R}$ that we have to find. We can solve the differential equation in eq. (3) by performing a change of variables operation. This will yield a new differential equation that resembles eq. (2), and reversing the change of variables operation will give us the solution to eq. (3).

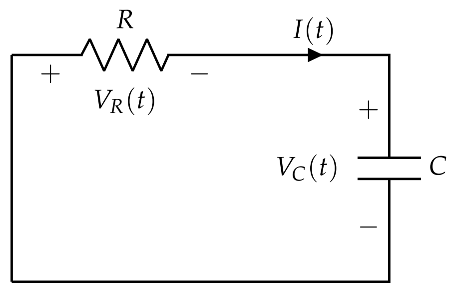


Figure 2: RC Circuit for part **1.a**. Note that the voltage source has been turned off (0 V) for this subpart, and the initial voltage on the capacitor is V_{DD}.

(a) Let's suppose that at $t = 0$, the capacitor is charged to a voltage V_{DD} ($V_C(0) = V_{DD}$). Let's also assume that $V(t) = 0$ for all $t \geq 0$ (voltage source is turned off). **Solve the differential equation for $V_C(t)$ for $t \geq 0$.**

Solution: Because $V(t) = 0$, the differential equation that we derived in the previous discussion (given in eq. (1)) simplifies to

$$RC\frac{\mathrm{d}V_C(t)}{\mathrm{d}t} + V_C(t) = 0 \tag{5}$$

Dividing both sides of the equation by RC, we arrive at

$$\frac{\mathrm{d}V_C(t)}{\mathrm{d}t} + \frac{1}{RC}V_C(t) = 0 \tag{6}$$

Moving the second term to the right-hand side, we have

$$\frac{\mathrm{d}V_C(t)}{\mathrm{d}t} = -\frac{1}{RC}V_C(t) \tag{7}$$

Notice that the form of this differential equation matches that of eq. (2) with $\lambda = -\frac{1}{RC}$ and $x(t) = V_C(t)$. We can now find the values of A and b as written in eq. (4).

To find A, we can use the initial condition ($V_C(0) = V_{DD}$) and follow the steps outlined in lecture:

$$\begin{aligned} V_{DD} &= V_C(0) && (8) \\ &= A\mathrm{e}^{b(0)} && (9) \\ &= A && (10) \end{aligned}$$

Now that we know the value of A, we can write that $V_C(t) = V_{DD}\mathrm{e}^{bt}$. The last task is to find b. We have already used the initial condition, so we must be able to find b from the differential equation. Plugging in our expression for $V_C(t)$ into the differential equation, we find

$$\begin{aligned} \frac{\mathrm{d}V_C(t)}{\mathrm{d}t} &= -\frac{1}{RC}V_C(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}V_{DD}\mathrm{e}^{bt} &= bV_{DD}\mathrm{e}^{bt} = bV_C(t) \\ &= -\frac{1}{RC}V_C(t) \\ \implies b &= -\frac{1}{RC}. \end{aligned}$$

In this case, we see the value of the remaining constant $b = -\frac{1}{RC}$, and our overall solution is

$$V_C(t) = V_{DD}\mathrm{e}^{-\frac{1}{RC}t} \tag{11}$$

A plot of the voltage curve will resemble figure 3.

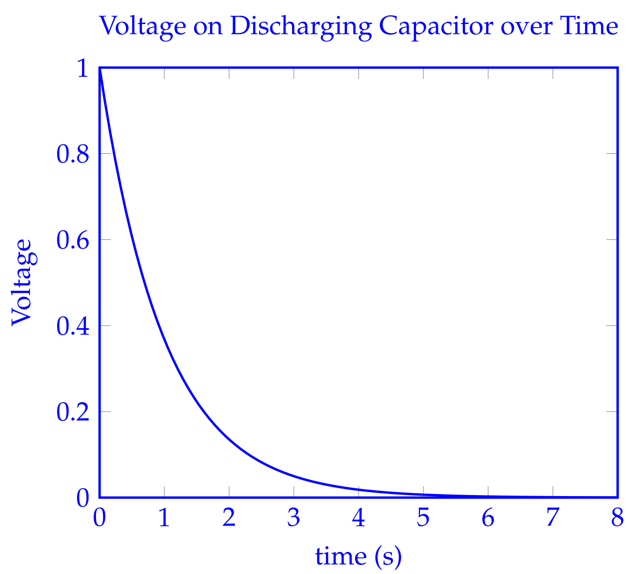


Figure 3: Plot of eq. (11) with $RC = 1$ and $V_{DD} = 1$

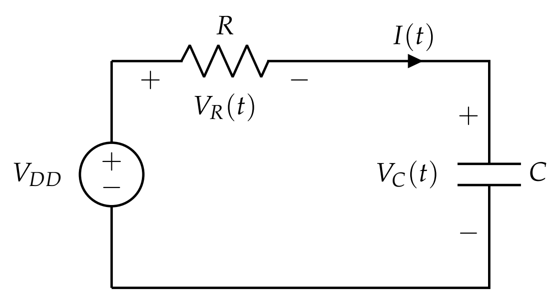


Figure 4: Circuit for part 1.b

(b) Now, let's suppose that we start with an uncharged capacitor $V_C(0) = 0$. We apply some constant voltage $V(t) = V_{DD}$ across the circuit for all $t \geq 0$. **Solve the differential equation for $V_C(t)$ for $t \geq 0$.**

Solution: Substituting $V(t) = V_{DD}$ into our solution from eq. (1):

$$RC\frac{\mathrm{d}V_C(t)}{\mathrm{d}t} + V_C(t) = V_{DD} \tag{12}$$

We want to arrange this equation to be in a form that we know how to solve:

$$\frac{\mathrm{d}}{\mathrm{d}t}V_C(t) = \frac{V_{DD} - V_C(t)}{RC} = -\frac{(V_C(t) - V_{DD})}{RC} \tag{13}$$

This form matches the differential equation we see in eq. (3), with $\lambda = -\frac{1}{RC}$ and $u = \frac{V_{DD}}{RC}$. We would like to apply a change of variables operation to turn this problem into one we already know how to solve, namely, the differential equation in eq. (2). Performing the change of variables operation will let us define a new differential equation in terms of a new function, which we will call $\widetilde{V}_C(t)$. Specifically, we want to achieve

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{V}_C(t) = \lambda \widetilde{V}_C(t) \tag{14}$$

Hence, we want to define $\widetilde{V}_C(t)$ so that $\frac{\mathrm{d}\widetilde{V}_C(t)}{\mathrm{d}t} = \frac{\mathrm{d}V_C(t)}{\mathrm{d}t}$ (i.e. the left hand sides of eq. (13) and eq. (14) are equal) and so that $\lambda \widetilde{V}_C(t) = -\frac{(V_C(t)-V_{DD})}{RC}$ (i.e. the right hand sides are equal). We can choose $\lambda = -\frac{1}{RC}$ which leaves us with $\widetilde{V}_C(t) = V_C(t) - V_{DD}$. This also satisfies the first condition that $\frac{\mathrm{d}\widetilde{V}_C(t)}{\mathrm{d}t} = \frac{\mathrm{d}V_C(t)}{\mathrm{d}t}$, so our change of variables here is valid. The new differential equation for $\widetilde{V}_C(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{V}_C(t) = -\frac{1}{RC}\widetilde{V}_C(t) \tag{15}$$

And so we get back almost the same differential equation as in the previous part, this time for $\widetilde{V}_C(t)$, with the only difference being that the initial condition changed! The new initial condition is $\widetilde{V}_C(0) = V_C(0) - V_{DD} = -V_{DD}$. And so, we can use that solution to get

$$\widetilde{V}_C(t) = \widetilde{V}_C(0)\mathrm{e}^{-\frac{1}{RC}t} = -V_{DD}\mathrm{e}^{-\frac{1}{RC}t}. \tag{16}$$

Finally, we need the solution in terms of $V_C(t)$ and not $\widetilde{V}_C(t)$, so we back-substitute:

$$\begin{aligned} V_C(t) &= V_{DD} + \widetilde{V}_C(t) && (17)\\ &= V_{DD} - V_{DD}\mathrm{e}^{-\frac{1}{RC}t} && (18)\\ &= V_{DD}(1 - \mathrm{e}^{-\frac{1}{RC}t}). && (19) \end{aligned}$$

A plot of the voltage curve will resemble figure 5.

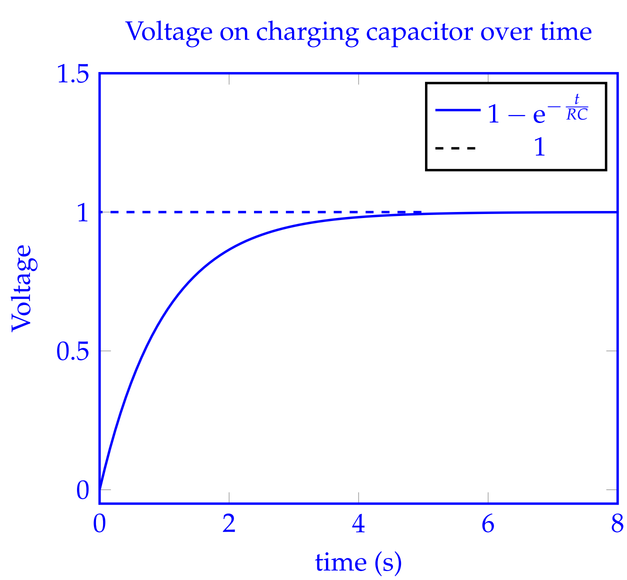


Figure 5: Plot of eq. (19) with $V_{DD} = 1$ and $RC = 1$

(c) We now want to combine the principles from the previous two subparts to understand the voltage waveform when a switch occurs at some time t. Specifically, suppose that at $t = 0$, $V(t) = 0\,\mathrm{V}$, $V_C(0) = V_{DD}$. Then, at some $t = t_{\text{switch}}$, the voltage source is turned on $V(t) = V_{DD}$ for $t \geq t_{\text{switch}}$. **Find the equation for the overall capacitor voltage as a function of time (for times before and after t_{switch}).**

Solution: The procedure here can be realized as a two step process:

i. Determine a function to model the behavior of the circuit until t_{switch} and the voltage across the capacitor at t_{switch}, i.e. $V_C(t_{\text{switch}})$

ii. Use $V_C(t_{\text{switch}})$ as an initial condition to find a function to model the behavior of the circuit after t_{switch}

We can write the breakdown of steps mathematically as a piecewise function:

$$V_C(t) = \begin{cases} V_{C,1}(t) & 0 \leq t < t_{\text{switch}} \\ V_{C,2}(t) & t \geq t_{\text{switch}} \end{cases} \tag{20}$$

for some functions $V_{C,1}(t)$ and $V_{C,2}(t)$. We can first find $V_{C,1}(t)$. We are told that $V_C(0) = 0$, so $V_{C,1}(0) = 0$. Up until t_{switch}, the behavior of the circuit is exactly the same as in part **1**.a. Therefore,

$$V_{C,1}(t) = V_{DD}\mathrm{e}^{-\frac{1}{RC}t} \tag{21}$$

The next step is to find the voltage across the capacitor at $t = t_{\text{switch}}$. To find this, we can compute $V_C(t_{\text{switch}}) = V_{DD}\mathrm{e}^{-\frac{t_{\text{switch}}}{RC}}$. This will serve as an initial condition to the next part, which is determining $V_{C,2}(t)$. An initial condition is not constrained to $t = 0$. An initial condition at an arbitrary t_{switch} can be thought of as a shift in the time axis, such that time now “starts” at $t = t_{\text{switch}}$ rather than $t = 0$.

At $t = t_{\text{switch}}$, the capacitor is charging, so it will behave similarly to the circuit in part **1**.b but with a different initial condition. Explicitly, the differential equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}V_{C,2}(t) = \frac{V_{DD} - V_{C,2}(t)}{RC};\ V_{C,2}(t_{\text{switch}}) = V_{DD}\mathrm{e}^{-\frac{t_{\text{switch}}}{RC}} \tag{22}$$

We can similarly define $\widetilde{V}_{C,2}(t) = V_{C,2}(t) - V_{DD}$. Note that $\widetilde{V}_{C,2}(t_{\text{switch}}) = V_{C,2}(t_{\text{switch}}) - V_{DD} = V_{DD}\left(\mathrm{e}^{-\frac{t_{\text{switch}}}{RC}} - 1\right)$. The new differential equation for $\widetilde{V}_{C,2}(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{V}_{C,2}(t) = -\frac{1}{RC}\widetilde{V}_{C,2}(t);\ \widetilde{V}_{C,2}(t_{\text{switch}}) = V_{DD}\left(\mathrm{e}^{-\frac{t_{\text{switch}}}{RC}} - 1\right) \tag{23}$$

We know the solution to eq. (23) will take the form of eq. (4). Since b did not rely on the initial condition, it will be the same as in part **1**.b (i.e. $b = -\frac{1}{RC}$). As before, we can use the initial condition to find A:

$$\widetilde{V}_{C,2}(t_{\text{switch}}) = A\mathrm{e}^{-\frac{t_{\text{switch}}}{RC}} = V_{DD}\left(\mathrm{e}^{-\frac{t_{\text{switch}}}{RC}} - 1\right) \implies A = V_{DD}\left(1 - \mathrm{e}^{\frac{t_{\text{switch}}}{RC}}\right) \tag{24}$$

Therefore,

$$\widetilde{V}_{C,2}(t) = V_{DD}\left(1 - \mathrm{e}^{\frac{t_{\text{switch}}}{RC}}\right)\mathrm{e}^{-\frac{t}{RC}} = V_{DD}\left(\mathrm{e}^{-\frac{t}{RC}} - \mathrm{e}^{-\frac{(t - t_{\text{switch}})}{RC}}\right) \tag{25}$$

and substituting back to find $V_{C,2}(t)$ gives us

$$V_{C,2}(t) = V_{DD}\left(1 + \mathrm{e}^{-\frac{t}{RC}} - \mathrm{e}^{-\frac{(t-t_{\text{switch}})}{RC}}\right) \tag{26}$$

And that's how to solve what happens when an RC circuit switches! A plot of the voltage curve will resemble figure 6.

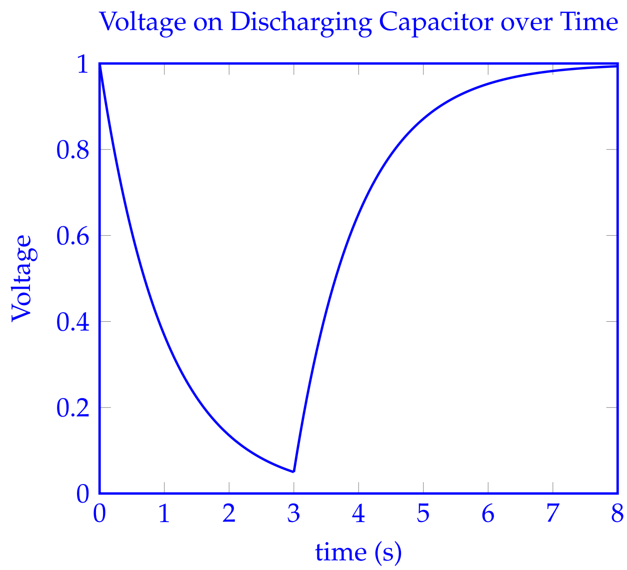


Figure 6: Plot of eq. (26) with $RC = 1$, $V_{DD} = 1$, and $t_{\text{switch}} = 3$

See an interactive demo here: https://www.desmos.com/calculator/gvjm36oo6j

2. Complex Algebra (Review)

(a) **Express the following values in polar forms:** -1, j, $-$j, $(\mathrm{j})^{\frac{1}{2}}$, **and** $(-\mathrm{j})^{\frac{1}{2}}$. Recall $\mathrm{j}^2 = -1$, and the complex conjugate of a complex number is denoted with a bar over the variable. The complex conjugate is defined as follows: for a complex number $z = x + \mathrm{j}y$, the complex conjugate $\overline{z} = x - \mathrm{j}y$.

Solution: Here, we review some basic properties of complex numbers and its rectangular and polar form: $z = x + \mathrm{j}y = |z|\mathrm{e}^{\mathrm{j}\theta}$, where $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$ and $\measuredangle z = \theta = \operatorname{atan2}(y, x)$. We can also write $x = |z|\cos(\theta)$, $y = |z|\sin(\theta)$.

A complex number can be represented in the following forms:

$$z = a + \mathrm{j}b = r\cos(\theta) + \mathrm{j}r\sin(\theta) = r\mathrm{e}^{\mathrm{j}\theta}, \tag{27}$$

where, $r = \sqrt{a^2 + b^2}$,$\measuredangle z = \operatorname{atan2}(b, a)$ and a, b are real numbers.

$$-1 = \mathrm{j}^2 = \mathrm{e}^{\mathrm{j}\pi} = \mathrm{e}^{-\mathrm{j}\pi} \tag{28}$$

$$\mathrm{j} = \mathrm{e}^{\mathrm{j}\frac{\pi}{2}} = \sqrt{-1} \tag{29}$$

$$-\mathrm{j} = -\mathrm{e}^{\mathrm{j}\frac{\pi}{2}} = \mathrm{e}^{-\mathrm{j}\frac{\pi}{2}} \tag{30}$$

$$(\mathrm{j})^{\frac{1}{2}} = (\mathrm{e}^{\mathrm{j}\frac{\pi}{2}})^{\frac{1}{2}} = \mathrm{e}^{\mathrm{j}\frac{\pi}{4}} = \frac{1+\mathrm{j}}{\sqrt{2}} \tag{31}$$

$$(-\mathrm{j})^{\frac{1}{2}} = (\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}})^{\frac{1}{2}} = \mathrm{e}^{-\mathrm{j}\frac{\pi}{4}} = \frac{1-\mathrm{j}}{\sqrt{2}} \tag{32}$$

(b) Represent $\sin(\theta)$ and $\cos(\theta)$ using complex exponentials. (*Hint:* Use Euler's identity $\mathrm{e}^{\mathrm{j}\theta} = \cos(\theta) + \mathrm{j}\sin(\theta)$.)

Solution: Note that we can use the fact that $\cos(x)$ is an even function, and $\sin(x)$ is an odd function. This gives us that:

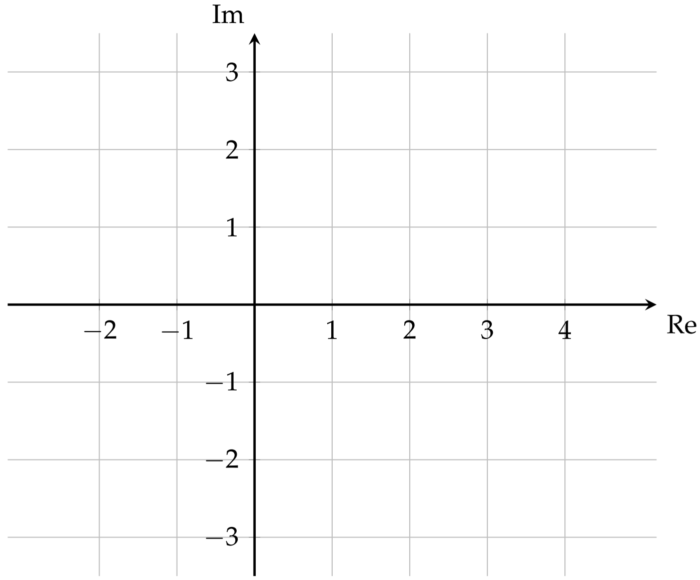
$$\begin{aligned}
\mathrm{e}^{\mathrm{j}\theta} &= \cos(\theta) + \mathrm{j}\sin(\theta) \\
\mathrm{e}^{-\mathrm{j}\theta} &= \cos(-\theta) + \mathrm{j}\sin(-\theta) \\
&= \cos(\theta) - \mathrm{j}\sin(\theta)
\end{aligned}$$

Solving this system of equations for $\cos(\theta)$ and $\sin(\theta)$ gives:

$$\sin(\theta) = \frac{\mathrm{e}^{\mathrm{j}\theta} - \mathrm{e}^{-\mathrm{j}\theta}}{2\mathrm{j}} \qquad \cos(\theta) = \frac{\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}\theta}}{2}$$

For the next parts, let $a = 1 - \mathrm{j}\sqrt{3}$ and $b = \sqrt{3} + \mathrm{j}$.

(c) Show the number a in complex plane, marking the distance from origin and angle with real axis.



Solution: The location of a in the complex plane is shown in Figure 7. The only two pieces of information we need are the magnitude and the phase, which is the polar coordinates interpretation. We could also use the (perhaps more familiar) x and y Cartesian coordinates.

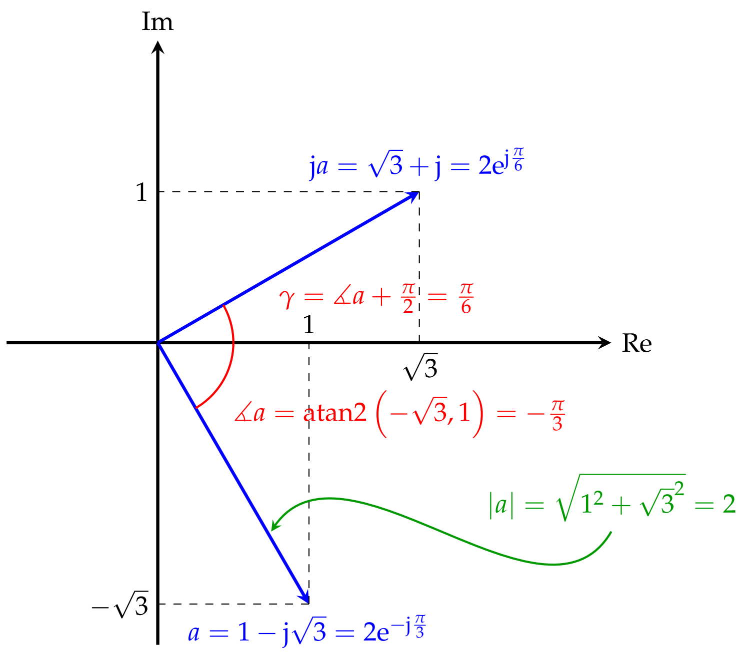


Figure 7: Complex numbers a and its rotated version b represented as vectors in the complex plane.

(d) Show that multiplying a with j is equivalent to rotating the complex number by $\frac{\pi}{2}$ or 90° in the complex plane.

Solution: Multiplying a by j:

$$\mathrm{j}a = \mathrm{e}^{\mathrm{j}\pi/2} \cdot 2\mathrm{e}^{-\mathrm{j}\pi/3} = 2\mathrm{e}^{\mathrm{j}\pi/6} = \sqrt{3} + \mathrm{j}$$

The rotation is demonstrated in the same complex plane plot (Figure 7), with a new angle $\gamma = \measuredangle a + \frac{\pi}{2}$.

(e) **(Practice)** For complex number $z = x + jy$ show that $|z| = \sqrt{z\bar{z}}$, where $\bar{z}$ is the complex conjugate of z.

Solution: We can follow the definition of complex conjugate and magnitude:

$$\sqrt{z\bar{z}} = \sqrt{(x + jy)(x - jy)} = \sqrt{x^2 + y^2} = |z| \tag{33}$$

(f) **(Practice)** Express a and b in polar form.

Solution: Following the definitions in part a):

$$|a| = 2$$
$$|b| = 2$$
$$\measuredangle a = -\frac{\pi}{3}$$
$$\measuredangle b = \frac{\pi}{6}$$

Hence:

$$a = 2e^{-j\frac{\pi}{3}} \qquad b = 2e^{j\frac{\pi}{6}}$$

(g) **(Practice)** Find ab, $a\bar{b}$, $\frac{a}{b}$, $a + \bar{a}$, $a - \bar{a}$, $\overline{ab}$, $\bar{a}\bar{b}$, and $(b)^{\frac{1}{2}}$.

Solution: We can evaluate these sequentially using the rules of complex algebra:

$$ab = 4 \cdot e^{-j\frac{\pi}{6}} = 2\sqrt{3} - 2j$$
$$a\bar{b} = 4 \cdot e^{-j\frac{\pi}{2}} = -4j$$
$$\frac{a}{b} = e^{-j\frac{\pi}{2}} = -j$$
$$a + \bar{a} = 2$$
$$a - \bar{a} = -2j\sqrt{3}$$
$$\overline{ab} = 2\sqrt{3} + 2j$$
$$\bar{a}\bar{b} = (1 + j\sqrt{3})(\sqrt{3} - j) = \sqrt{3} + \sqrt{3} + j(3 - 1) = 2\sqrt{3} + 2j$$
$$(b)^{\frac{1}{2}} = \sqrt{2}e^{j\frac{\pi}{12}}$$

Note the following: $a + \bar{a}$ is a purely real number. $a - \bar{a}$ is a purely imaginary number. And, $\overline{ab} = \bar{a}\bar{b}$.

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