Discussion 1B

For this discussion, Note 1 and Note 2 as well as the corresponding lectures are prerequisite materials to review and have available while completing the problems.

1. Analyzing an RC Circuit with a Sinusoidal Source (Adapted from Hambley Example 4.4)

Assume you are given the following circuit, where the capacitor is initially charged so that $v_C(t) = 1V$.

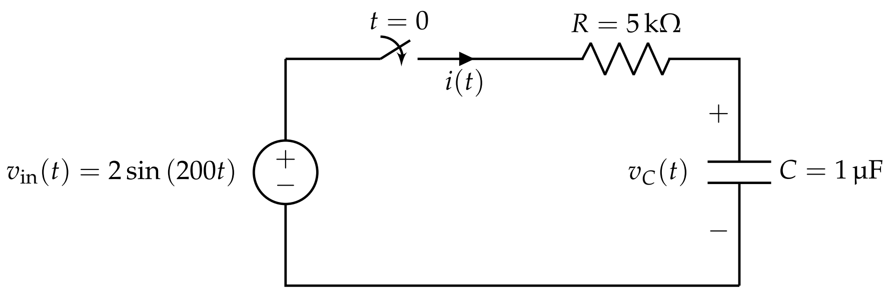


Figure 1

(a) **Set up a differential equation for the voltage $v_C(t)$ across the capacitor in the form:**

$$\frac{dv_C(t)}{dt} = \lambda v_C(t) + u(t) \tag{1}$$

Solution: Let's start by writing the current equation for $t > 0$ using KCL on the node that connects the resistor to the capacitor:

$$i_R(t) = i_C(t) \tag{2}$$

$$\frac{v_{in}(t) - v_C(t)}{R} = C\frac{dv_C(t)}{dt} \tag{3}$$

$$\frac{dv_C(t)}{dt} = \frac{-1}{RC}v_C(t) + \frac{v_{in}(t)}{RC} \tag{4}$$

where $\lambda = -\frac{1}{RC} = -\frac{1}{5\times10^{-3}} = -200$ and $u(t) = \frac{v_{in}(t)}{RC} = 400\sin(200t)$

(b) **What is the initial condition of $v_C(t)$? In other words, what is $v(0)$?**

Solution: We are told in the problem that $v_C(0) = 1\,\text{V}$. If we were solving for current or using a more complicated setup, we may have to consider all of the components' steady-state properties.

(c) **Solve for the voltage $v_C(t)$ through the capacitor. Also, identify the transient response (homogeneous solution) and the forced response (particular solution) of $v_C(t)$.** You may directly use the fact that the solution to a differential equation in the same form as Equation 1 is:

$$v_C(t) = v(t_0)e^{\lambda(t-t_0)} + e^{\lambda t}\int_{t_0}^{t} e^{-\lambda\theta}u(\theta)\,d\theta \tag{5}$$

(HINT: The following integral might be useful:

$$\int e^{at}\sin(bt) = \frac{1}{b^2+a^2}e^{at}(b\sin(bt) - a\cos(bt)) \tag{6}$$

)

Solution: Now that we have our differential equation in the standard form, we can use the general form of the solution to a first-order differential equation to solve for $i(t)$. Recall that the solution is:

$$v_C(t) = v(t_0)e^{\lambda(t-t_0)} + e^{\lambda t}\int_{t_0}^{t} e^{-\lambda\theta}u(\theta)\,d\theta \tag{7}$$

where $t_0 = 0$ in this specific case. So using out differential equation and plugging in λ and $u(t)$, we get:

$$v_C(t) = e^{-\frac{1}{RC}t} + e^{-\frac{1}{RC}t}\int_0^t e^{\frac{1}{RC}t'} 400\sin(200t')\,dt' \tag{8}$$

$$v_C(t) = e^{-200t} + 400e^{-\frac{1}{RC}t}\int_0^t e^{\frac{1}{RC}t'}\sin(200t')\,dt' \tag{9}$$

From here, we can apply Equation 6 from the hint and substituting values for $R = 5\,\mathrm{k\Omega}$ and $C = 1\,\mu\mathrm{F}$:

$$v_C(t) = e^{-200t} + 400e^{-\frac{1}{RC}t}\left[\frac{1}{\frac{1}{RC}^2 + 200^2}e^{\frac{1}{RC}t'}\left(200\sin(200t') - \frac{1}{RC}\cos(200t')\right)\right]_0^t \tag{10}$$

$$v_C(t) = e^{-200t} + e^{-200t}\frac{400}{200^2+200^2}\left[e^{200t'}\left(200\sin(200t') - 200\cos(200t')\right)\right]_0^t \tag{11}$$

$$v_C(t) = e^{-200t} + e^{-200t}\frac{400}{200^2+200^2}\left[\left(e^{200t}\left(200\sin(200t) - 200\cos(200t)\right)\right) - (-200)\right] \tag{12}$$

$$v_C(t) = e^{-200t} + e^{-200t}\frac{400}{200^2+200^2}e^{200t}\left(200\sin(200t) - 200\cos(200t)\right) + \frac{400\cdot 200}{200^2+200^2}e^{-200t} \tag{13}$$

$$v_C(t) = \left(1 + \frac{400\cdot 200}{200^2+200^2}\right)e^{-200t} + \left(\sin(200t) - \cos(200t)\right) \tag{14}$$

$$v_C(t) = 2e^{-200t} + \left(\sin(200t) - \cos(200t)\right)\,\mathrm{V} \tag{15}$$

where the transient response is $2e^{-200t}$ (goes to 0 over time) and the forced response is $\sin(200t) - \cos(200t)$.

(d) **(OPTIONAL) Solve for the current $i(t)$ through the circuit.**

Solution: There are two ways to go about solving for $i_C(t)$:

(1) You can solve for the voltage by either setting up a differential equation for $i_C(t)$ and solving the differential equation or

(2) use the solution from part (a) and the IV relationship for a capacitor. For this problem, we are simply going to use our solution from part (a) and plug it into the voltage of a capacitor in terms of current:

$$i_C(t) = C\frac{dv_C}{dt} \tag{16}$$

$$i_C(t) = C\frac{d(2e^{-200t} + (\sin(200t) - \cos(200t)))}{dt} \tag{17}$$

$$i_C(t) = -400e^{-200t} + (200\sin(200t) + 200\cos(200t))\,\mu\mathrm{A} \tag{18}$$

2. Complex Algebra (Review)

(a) **Express the following values in polar forms:** -1, j, $-$j, $(\mathrm{j})^{\frac{1}{2}}$**, and** $(-\mathrm{j})^{\frac{1}{2}}$**.** Recall $\mathrm{j}^2 = -1$, and the complex conjugate of a complex number is denoted with a bar over the variable. The complex conjugate is defined as follows: for a complex number $z = x + \mathrm{j}y$, the complex conjugate $\overline{z} = x - \mathrm{j}y$.

Solution: Here, we review some basic properties of complex numbers and its rectangular and polar form: $z = x + \mathrm{j}y = |z|\mathrm{e}^{\mathrm{j}\theta}$, where $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$ and $\measuredangle z = \theta = \mathrm{atan2}\,(y, x)$. We can also write $x = |z|\cos(\theta)$, $y = |z|\sin(\theta)$.

A complex number can be represented in the following forms:

$$z = a + \mathrm{j}b = r\cos(\theta) + \mathrm{j}r\sin(\theta) = r\mathrm{e}^{\mathrm{j}\theta}, \tag{19}$$

where, $r = \sqrt{a^2 + b^2}$,$\measuredangle z = \mathrm{atan2}(b, a)$ and a, b are real numbers.

$$-1 = \mathrm{j}^2 = \mathrm{e}^{\mathrm{j}\pi} = \mathrm{e}^{-\mathrm{j}\pi} \tag{20}$$

$$\mathrm{j} = \mathrm{e}^{\mathrm{j}\frac{\pi}{2}} = \sqrt{-1} \tag{21}$$

$$-\mathrm{j} = -\mathrm{e}^{\mathrm{j}\frac{\pi}{2}} = \mathrm{e}^{-\mathrm{j}\frac{\pi}{2}} \tag{22}$$

$$(\mathrm{j})^{\frac{1}{2}} = (\mathrm{e}^{\mathrm{j}\frac{\pi}{2}})^{\frac{1}{2}} = \mathrm{e}^{\mathrm{j}\frac{\pi}{4}} = \frac{1+\mathrm{j}}{\sqrt{2}} \tag{23}$$

$$(-\mathrm{j})^{\frac{1}{2}} = (\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}})^{\frac{1}{2}} = \mathrm{e}^{-\mathrm{j}\frac{\pi}{4}} = \frac{1-\mathrm{j}}{\sqrt{2}} \tag{24}$$

(b) Represent $\sin(\theta)$ and $\cos(\theta)$ using complex exponentials. (*Hint:* Use Euler's identity $\mathrm{e}^{\mathrm{j}\theta} = \cos(\theta) + \mathrm{j}\sin(\theta)$.)

Solution: Note that we can use the fact that $\cos(x)$ is an even function, and $\sin(x)$ is an odd function. This gives us that:

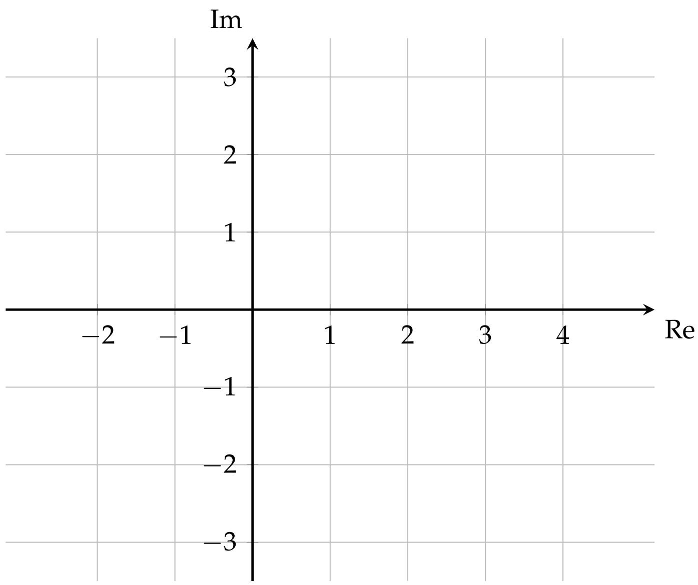
$$\begin{aligned}
\mathrm{e}^{\mathrm{j}\theta} &= \cos(\theta) + \mathrm{j}\sin(\theta) \\
\mathrm{e}^{-\mathrm{j}\theta} &= \cos(-\theta) + \mathrm{j}\sin(-\theta) \\
&= \cos(\theta) - \mathrm{j}\sin(\theta)
\end{aligned}$$

Solving this system of equations for $\cos(\theta)$ and $\sin(\theta)$ gives:

$$\sin(\theta) = \frac{\mathrm{e}^{\mathrm{j}\theta} - \mathrm{e}^{-\mathrm{j}\theta}}{2\mathrm{j}} \qquad \cos(\theta) = \frac{\mathrm{e}^{\mathrm{j}\theta} + \mathrm{e}^{-\mathrm{j}\theta}}{2}$$

For the next parts, let $a = 1 - \mathrm{j}\sqrt{3}$ and $b = \sqrt{3} + \mathrm{j}$.

(c) Show the number a in complex plane, marking the distance from origin and angle with real axis.



Solution: The location of a in the complex plane is shown in Figure 2. The only two pieces of information we need are the magnitude and the phase, which is the polar coordinates interpretation. We could also use the (perhaps more familiar) x and y Cartesian coordinates.

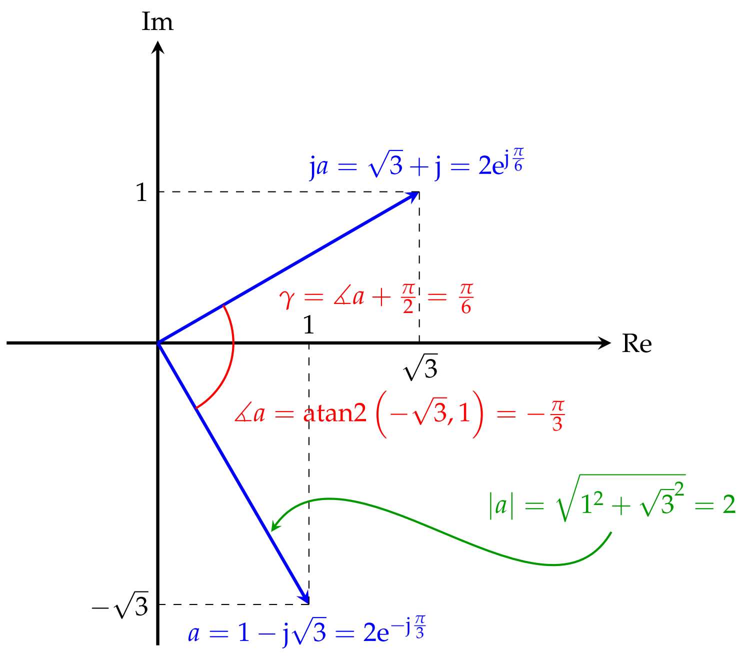


Figure 2: Complex numbers a and its rotated version b represented as vectors in the complex plane.

(d) Show that multiplying a with j is equivalent to rotating the complex number by $\frac{\pi}{2}$ or 90° in the complex plane.

Solution: Multiplying a by j:

$$\mathrm{j}a = e^{\mathrm{j}\pi/2} \cdot 2e^{-\mathrm{j}\pi/3} = 2e^{\mathrm{j}\pi/6} = \sqrt{3} + \mathrm{j}$$

The rotation is demonstrated in the same complex plane plot (Figure 2), with a new angle $\gamma = \measuredangle a + \frac{\pi}{2}$.

(e) **(Practice)** For complex number $z = x + jy$ show that $|z| = \sqrt{z\bar{z}}$, where $\bar{z}$ is the complex conjugate of z.

Solution: We can follow the definition of complex conjugate and magnitude:

$$\sqrt{z\bar{z}} = \sqrt{(x + jy)(x - jy)} = \sqrt{x^2 + y^2} = |z| \tag{25}$$

(f) **(Practice)** Express a and b in polar form.

Solution: Following the definitions in part a):

$$|a| = 2$$
$$|b| = 2$$
$$\measuredangle a = -\frac{\pi}{3}$$
$$\measuredangle b = \frac{\pi}{6}$$

Hence:

$$a = 2e^{-j\frac{\pi}{3}} \qquad b = 2e^{j\frac{\pi}{6}}$$

(g) **(Practice)** Find ab, $a\bar{b}$, $\frac{a}{b}$, $a + \bar{a}$, $a - \bar{a}$, $\overline{ab}$, $\bar{a}\bar{b}$, and $(b)^{\frac{1}{2}}$.

Solution: We can evaluate these sequentially using the rules of complex algebra:

$$ab = 4 \cdot e^{-j\frac{\pi}{6}} = 2\sqrt{3} - 2j$$
$$a\bar{b} = 4 \cdot e^{-j\frac{\pi}{2}} = -4j$$
$$\frac{a}{b} = e^{-j\frac{\pi}{2}} = -j$$
$$a + \bar{a} = 2$$
$$a - \bar{a} = -2j\sqrt{3}$$
$$\overline{ab} = 2\sqrt{3} + 2j$$
$$\bar{a}\bar{b} = (1 + j\sqrt{3})(\sqrt{3} - j) = \sqrt{3} + \sqrt{3} + j(3 - 1) = 2\sqrt{3} + 2j$$
$$(b)^{\frac{1}{2}} = \sqrt{2}e^{j\frac{\pi}{12}}$$

Note the following: $a + \bar{a}$ is a purely real number. $a - \bar{a}$ is a purely imaginary number. And, $\overline{ab} = \bar{a}\bar{b}$.

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