

Discussion 1A

The material from both [Lecture 1](#) and [Note 1](#) are important prerequisites for this worksheet. Please review and have them on hand while doing the problems.

1. Current, Power, and Energy for a Capacitance (Hambley Example 3.3)

Suppose that the voltage waveform shown in Figure 1 is applied to a $10\text{-}\mu\text{F}$ capacitance.

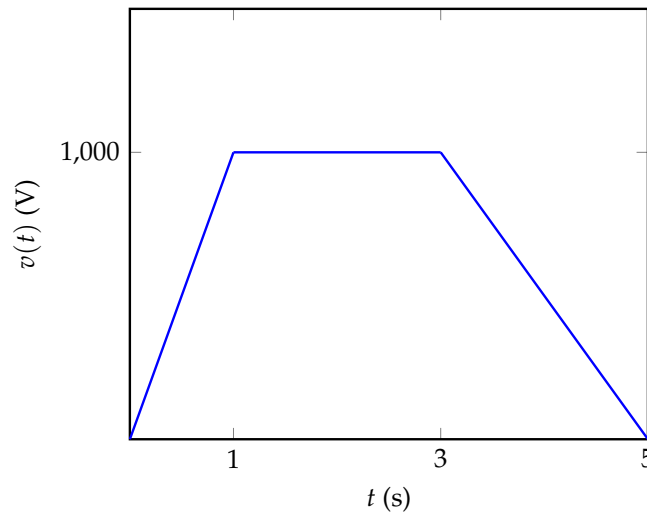


Figure 1: Plot of $v(t)$

Find and plot the current, the power delivered, and the energy stored for time between 0 and 5 s.

Solution:

First, we write expressions for the voltage as a function of time:

$$v(t) = \begin{cases} 1000t \text{ V} & 0 < t < 1 \\ 1000 \text{ V} & 1 < t < 3 \\ 500(5-t) \text{ V} & 3 < t < 5 \end{cases}$$

Then, using the equation

$$i(t) = C \frac{dv(t)}{dt} \quad (1)$$

We can obtain expressions for the current

$$i(t) = \begin{cases} 10 \times 10^{-3} \text{ A} & 0 < t < 1 \\ 0 \text{ A} & 1 < t < 3 \\ -5 \times 10^{-3} \text{ A} & 3 < t < 5 \end{cases}$$

Using the equation

$$p(t) = v(t)i(t) \quad (2)$$

We can obtain expressions for power

$$p(t) = \begin{cases} 10t \text{ W} & 0 < t < 1 \\ 0 \text{ W} & 1 < t < 3 \\ 2.5(t-5) \text{ W} & 3 < t < 5 \end{cases}$$

Lastly, using the equation

$$E(t) = \frac{1}{2} C v^2(t) \quad (3)$$

$$E(t) = \begin{cases} 5t^2 \text{ J} & 0 < t < 1 \\ 5 \text{ J} & 1 < t < 3 \\ 1.25(5-t)^2 \text{ J} & 3 < t < 5 \end{cases}$$

The plots for $i(t)$, $p(t)$, and $E(t)$ are as follows

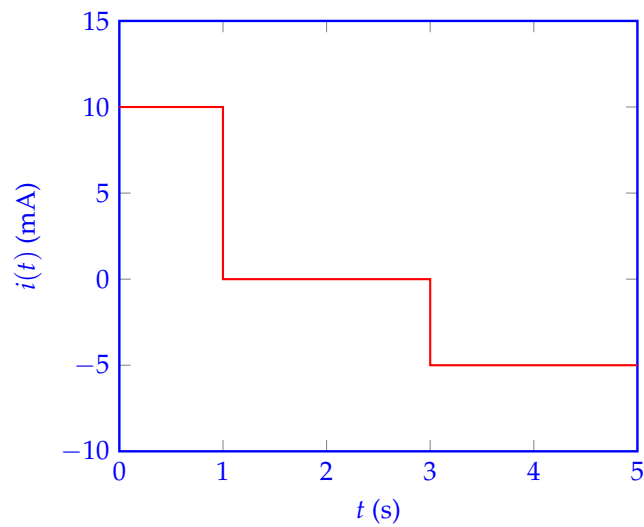


Figure 2: Plot of $i(t)$

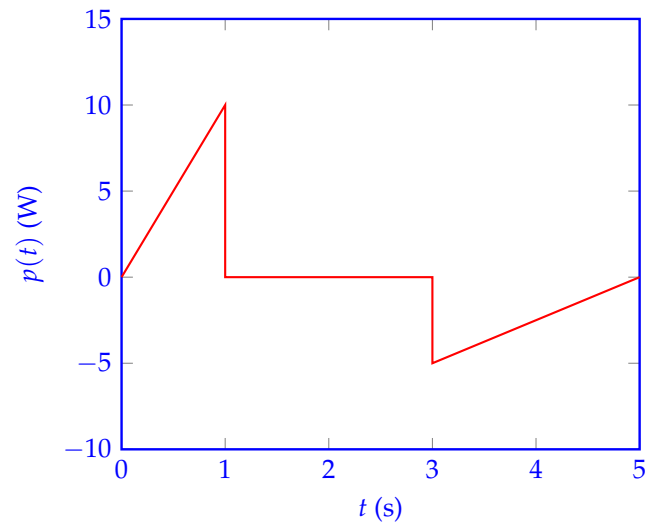


Figure 3: Plot of $p(t)$

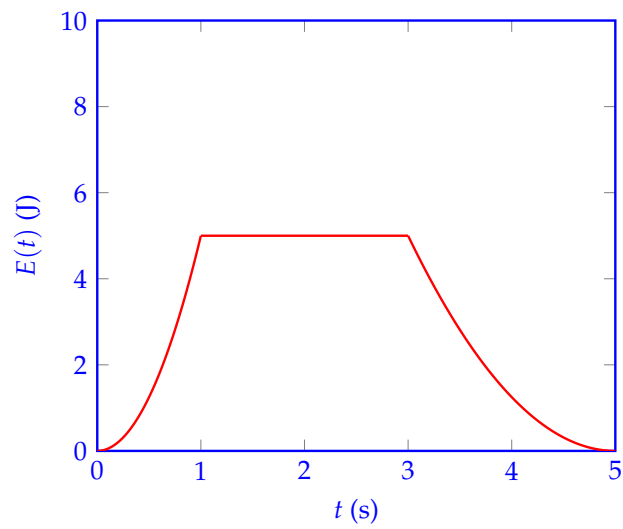


Figure 4: Plot of $E(t)$

2. Determining Voltage for a Capacitance Given Current (Hambley Example 3.2)

After t_0 the current in a $0.1 \mu\text{F}$ capacitor is given by

$$i(t) = 0.5 \sin 10^4 t \quad (4)$$

(The argument of the sin function is in radians.) The initial charge on the capacitor is $q(0) = 0$.

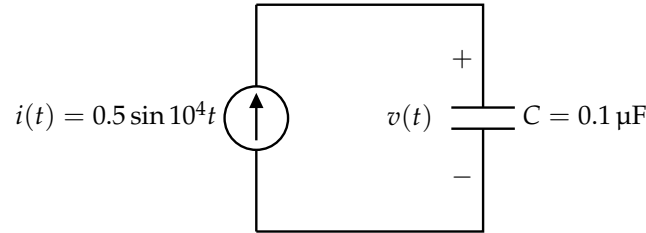


Figure 5: Example Circuit

Plot $i(t)$, $q(t)$, and $v(t)$ to scale versus time.

Solution:

Method 1 (solving for voltage first): Recall the equation relating the current and voltage of a capacitor:

$$i(t) = C \frac{dv(t)}{dt} \quad (5)$$

Since we intend to solve for voltage first, we rearrange the equation accordingly and integrate:

$$dv(t) = \frac{1}{C} i(t) dt \quad (6)$$

$$\int_{v(0)}^{v(t)} dv(t) = \int_0^t \frac{1}{C} i(t) dt \quad (7)$$

$$v(t) = \frac{1}{C} \int_0^t i(t) dt + v(0) \quad (8)$$

$$= \frac{1}{10^{-7}} \int_0^t 0.5 \sin 10^4 t dt + 0 \quad (9)$$

$$= \frac{1}{10^{-7}} \times -0.5 \times \left[10^{-4} \cos 10^4 t \right] \Big|_0^t \quad (10)$$

$$= \frac{1}{10^{-7}} \times 0.5 \times 10^{-4} \left[1 - \cos 10^4 t \right] \quad (11)$$

$$= 500 \left[1 - \cos 10^4 t \right] \quad (12)$$

The fact that $v(0)$ was zero came from the relationship $q(t) = Cv(t)$. Next, to find $q(t)$ we simply use this very relationship and multiply by capacitance.

$$q(t) = 0.5 \times 10^{-4} \left[1 - \cos 10^4 t \right] \quad (13)$$

Finally, we plot all three noted quantities (plots following method 2 solution).

Method 2 (solving for charge first): Recall the following equation

$$q(t) = \int_0^t i(t)dt + q(0) \quad (14)$$

We can directly use this equation to solve for the charge on the capacitor.

$$q(t) = \int_0^t i(t)dt + q(0) \quad (15)$$

$$= \int_0^t 0.5 \sin 10^4 t dt \quad (16)$$

$$= -0.5 \times \left[10^{-4} \cos 10^4 t \right] \Big|_0^t \quad (17)$$

$$= 0.5 \times 10^{-4} \left[1 - \cos 10^4 t \right] \quad (18)$$

Using the relationship between voltage and charge of a capacitor, we can solve for $v(t)$

$$v(t) = \frac{q(t)}{C} = \frac{q(t)}{10^{-7}} = 500 \left[1 - \cos 10^4 t \right] \quad (19)$$

Here are the plots of $i(t)$, $q(t)$, and $v(t)$

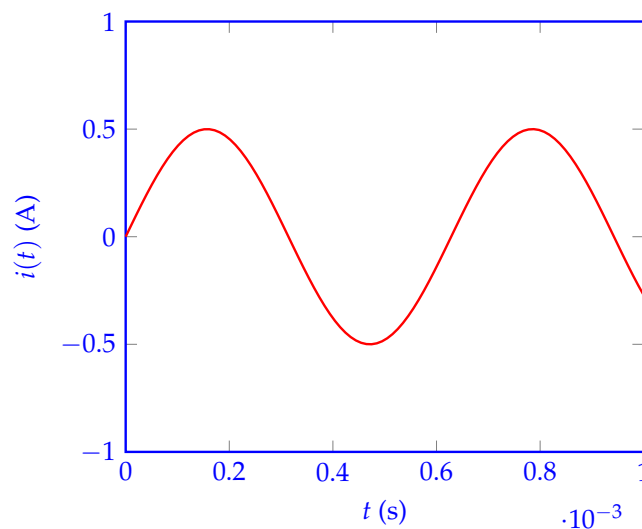


Figure 6: Plot of $i(t)$

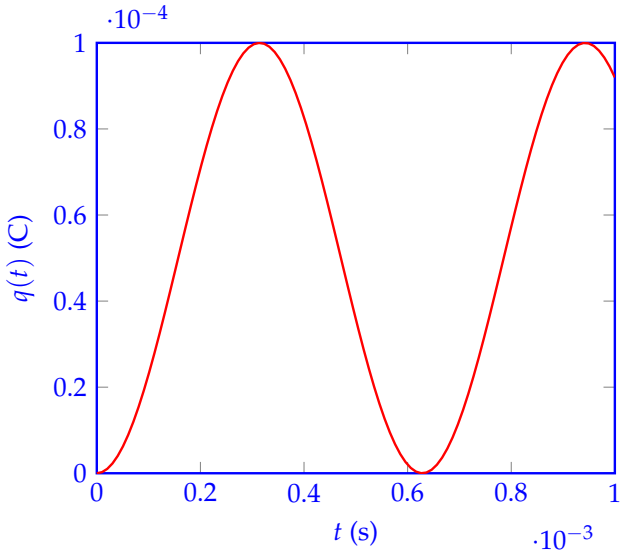


Figure 7: Plot of $q(t)$

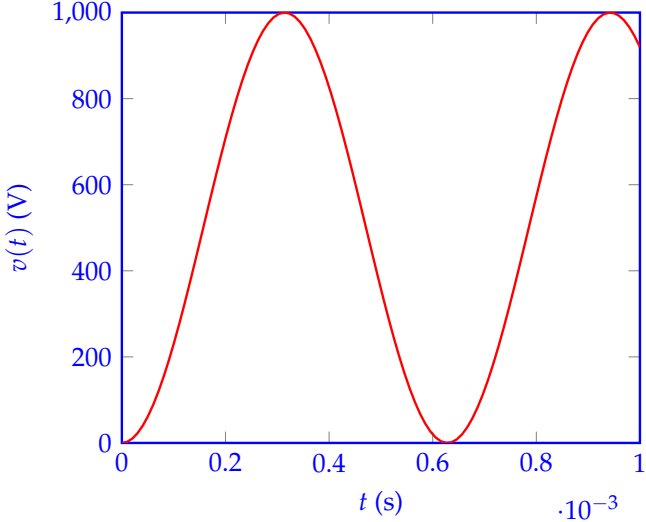


Figure 8: Plot of $v(t)$

3. Analyzing an RC Circuit with a Constant Source (Adapted from Hambley Example 4.4)

Assume you are given the following circuit, where the capacitor is initially charged so that $v_C(t) = 1V$.

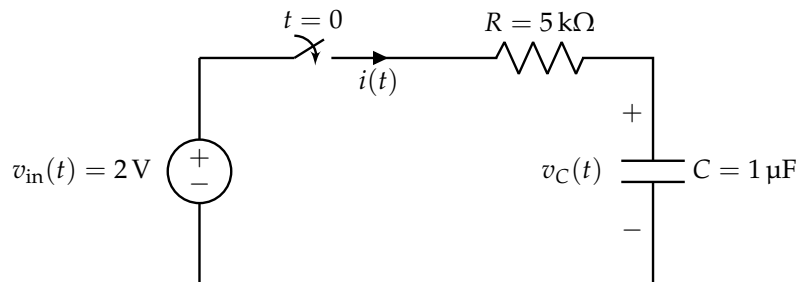


Figure 9

- (a) Set up a differential equation for the voltage $v_C(t)$ across the capacitor in the form:

$$\frac{dv_C(t)}{dt} = \lambda v_C(t) + u(t) \quad (20)$$

Solution: Let's start by writing the current equation for $t > 0$ using KCL on the node that connects the resistor to the capacitor:

$$i_R(t) = i_C(t) \quad (21)$$

$$\frac{v_{in}(t) - v_C(t)}{R} = C \frac{dv_C(t)}{dt} \quad (22)$$

$$\frac{dv_C(t)}{dt} = \frac{-1}{RC} v_C(t) + \frac{v_{in}(t)}{RC} \quad (23)$$

where $\lambda = -\frac{1}{RC} = -\frac{1}{5 \times 10^{-3}} = -200$ and $u(t) = \frac{v_{in}(t)}{RC} = 400$

- (b) Solve for the voltage $v_C(t)$ using the homogeneous and particular solution method. (HINT: Recall, that $v_C(t)$ is composed of a homogeneous and a particular solution. First, use your work from the previous part to find the homogeneous solution form. Then, to solve for both the unknown coefficient and the particular solution, find the initial condition of $v_C(t)$ using steady-state properties. As an extra check, you can always plug your final solution back into the differential equation to confirm.)

Solution: Part 1: Homogeneous Solution

From the notes and lecture, we know that the homogeneous solution $v_h(t)$ has the following form:

$$v_h(t) = Ae^{b(t-t_0)} = \quad (24)$$

$$= Ae^{bt} \quad (25)$$

$$= Ae^{\lambda t} \quad (26)$$

$$= Ae^{-200t} \quad (27)$$

where $b = \lambda = -200$ and $t_0 = 0$. We will determine the value of A when we find the initial condition for the circuit.

Part 2: Particular Solution

Although, we now have a family of possible solutions that can potentially describe this circuit (i.e. $v_C(t) = Ae^{-200t} + v_p(t)$), we want to identify the particular solution $v_p(t)$ that describes the dynamic of this capacitor circuit exactly.

To do that, we notice that the homogeneous or transient part of the solution will eventually die out *a long time in the future* (i.e. as $t \rightarrow \infty$). Therefore, we $v_p(t)$ is simply the voltage across the capacitor at some time "well into the future when the transient voltage has died out". In other words, it is the steady-state voltage.

The following circuit diagram shows the circuit in DC steady-state (the capacitor is replaced with an open circuit):

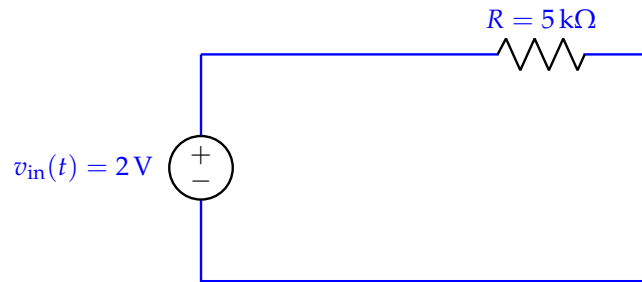


Figure 10

Since no current flows through the resistor (due to the open circuit), the voltage across the capacitor (represented by the open circuit) in steady-state must be the input voltage. Thus, the particular solution is:

$$v_p(t) = v_{in}(t) = 2V \quad (28)$$

Thus, our solution currently looks like:

$$v_C(t) = Ae^{-200t} + 2 \quad (29)$$

We are almost done! All that is left is to...

Part 3: Solve for A

Now, we have a solution for $v_C(t)$ with only one unknown. We also have one additional piece of information: the initial condition! Using it, we can solve for A and find that $v_C(t)$ is given by:

$$v_C(0) = 1 \quad (30)$$

$$Ae^{-200 \cdot 0} + 2 = 1 \quad (31)$$

$$A + 2 = 1 \quad (32)$$

$$A = -1 \quad (33)$$

Thus, the overall solution is:

$$v_C(t) = -e^{-200t} + 2 \quad (34)$$

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