1 Inner Products

An **inner product** $\langle \cdot, \cdot \rangle$ on a vector space V over $\mathbb{R}$ is a function that takes in two vectors and outputs a scalar, such that $\langle \cdot, \cdot \rangle$ is symmetric, linear, and positive-definite.

- Symmetry: $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
- Scaling: $\langle c\vec{u}, \vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$ and $\langle \vec{u}, c\vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$
- Additivity: $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ and $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$
- Positive-definite: $\langle \vec{u}, \vec{u} \rangle \geq 0$ with $\langle \vec{u}, \vec{u} \rangle = 0$ if and only if $\vec{u} = \vec{0}$

For two vectors, $\vec{u}, \vec{v} \in \mathbb{R}^n$, the standard inner product is $\langle \vec{u}, \vec{v} \rangle = \vec{u}^T\vec{v}$. We define the **norm**, or the magnitude, of a vector $\vec{v}$ to be $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{\vec{v}^T\vec{v}}$. For any non-zero vector, we can *normalize*, i.e., set its magnitude to 1 while preserving its direction, by dividing the vector by its norm $\frac{\vec{v}}{\|\vec{v}\|}$.

Orthogonality and Orthonormality

The inner product lets us define the angle between two vectors through the equation

$$\langle \vec{u}, \vec{v} \rangle = \|\vec{u}\|\|\vec{v}\| \cos\theta \tag{1}$$

Notice that if the angle θ between two vectors is $\pm 90°$, the inner product $\langle \vec{u}, \vec{v} \rangle = 0$.

Therefore, we define two vectors $\vec{u}$ and $\vec{v}$ to be **orthogonal** to each other if $\langle \vec{u}, \vec{v} \rangle = \vec{u}^T\vec{v} = 0$. A set of vectors is orthogonal if any two vectors in this set are orthogonal to each other.

Furthermore, we define two vectors $\vec{u}$ and $\vec{v}$ to be **orthonormal** to each other if they are orthogonal to each other and their norms are 1. A set of vectors is orthonormal if any two vectors in this set are orthogonal to each other and every vector has a norm of 1. In fact, for any two vectors $\vec{u}$ and $\vec{v}$ in an orthonormal set,

$$\langle \vec{u}, \vec{v} \rangle = \vec{u}^T\vec{v} = \begin{cases} 1, & \text{if } \vec{u} = \vec{v} \\ 0, & \text{otherwise} \end{cases}.$$

Unitary Matrices

An **orthogonal** or **unitary** matrix is a square matrix whose columns are orthonormal with respect to the inner product. To avoid any confusion, we will often refer to these matrices as **orthonormal matrices.**

$$U = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{bmatrix}, \qquad \vec{u}_i^T\vec{u}_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

Note that $U^TU = UU^T = I$, so the inverse of a unitary matrix is its transpose $U^{-1} = U^T$.

Since the columns of a unitary matrix are orthonormal vectors, we can interpret these matrices as "rotation" and "reflection" matrices of the standard axes. This also implies that $\|U\vec{v}\| = \|\vec{v}\|$ for any vector $\vec{v}$.

2 Spectral Theorem

Let A be an $n \times n$ **symmetric** matrix with real entries. Then the following statements will be true.

1. All eigenvalues of A are real.
2. A has n linearly independent eigenvectors $\in \mathbb{R}^n$.
3. A has orthogonal eigenvectors, i.e., $A = V\Lambda V^{-1} = V\Lambda V^T$, where Λ is a diagonal matrix and V is an orthonormal matrix. We say that A is orthogonally diagonalizable.

Recall that a matrix A is symmetric if $A = A^T$. Furthermore, if A is of the form $B^T B$ for some arbitrary matrix B, then all of the eigenvalues of A are non-negative, i.e., $\lambda \geq 0$.

a) Prove the following: All eigenvalues of a symmetric matrix A are real.

Hint: Let $(\lambda, \vec{v})$ be an eigenvalue/vector pair. Then $A\vec{v} = \lambda\vec{v}$ and take the complex conjugate and transpose of both sides. Try to show that $\overline{\lambda} = \lambda$.

Answer

Let λ be an eigenvalue of A with corresponding eigenvector $\vec{v}$.

$$A\vec{v} = \lambda\vec{v}$$

Then we can take the complex conjugate of both sides and use the fact that A is real.

$$A\overline{\vec{v}} = \overline{\lambda}\overline{\vec{v}}$$

Now let's take the transpose of both sides and use the fact that A is symmetric.

$$\overline{\vec{v}}^T A = \overline{\lambda}\overline{\vec{v}}^T$$

Let's right multiply by $\vec{v}$, to see that

$$\overline{\vec{v}}^T A\vec{v} = \overline{\lambda}\overline{\vec{v}}^T \vec{v}$$

Applying the eigenvector property once more,

$$\lambda\overline{\vec{v}}^T \vec{v} = \overline{\lambda}\overline{\vec{v}}^T \vec{v}$$

$\overline{\vec{v}}^T \vec{v}$ must be greater than 0 since $\vec{v}$ is an eigenvector and cannot be $\vec{0}$. Therefore, the only possibility is $\lambda = \overline{\lambda}$ which implies that λ is real.

b) Prove the following: For any symmetric matrix A, any two eigenvectors corresponding to distinct eigenvalues of A are orthogonal.

Hint: Let $\vec{v}_1$ and $\vec{v}_2$ be eigenvectors of A with eigenvalues $\lambda_1 \neq \lambda_2$.

$$A\vec{v}_1 = \lambda_1\vec{v}_1$$
$$A\vec{v}_2 = \lambda_2\vec{v}_2$$

Take the transpose of the second equation and show that $\lambda_1\langle\vec{v}_1, \vec{v}_2\rangle = \lambda_2\langle\vec{v}_1, \vec{v}_2\rangle$.

Answer

Let's apply the hint by taking the transpose of the second equation.

$$A\vec{v}_1 = \lambda_1\vec{v}_1$$
$$\vec{v}_2^T A^T = \lambda_2\vec{v}_2^T$$

Since A is symmetric, $A^T = A$ and we can left-multiply the first equation by $\vec{v}_2^T$ and right-multiply the second equation by $\vec{v}_2$ to say

$$\vec{v}_2^T A\vec{v}_1 = \lambda_1\vec{v}_2^T\vec{v}_1$$
$$\vec{v}_2^T A\vec{v}_1 = \lambda_2\vec{v}_2^T\vec{v}_1$$

This tells us that $\lambda_1\langle\vec{v}_1,\vec{v}_2\rangle = \lambda_2\langle\vec{v}_1,\vec{v}_2\rangle$ meaning

$$(\lambda_1 - \lambda_2)\langle\vec{v}_1,\vec{v}_2\rangle = 0$$

The only way this equation can be satisfied when $\lambda_1 \neq \lambda_2$ is for $\langle\vec{v}_1,\vec{v}_2\rangle$ to be zero. Therefore, $\vec{v}_1$ and $\vec{v}_2$ must be orthogonal to each other

c) Prove the following: For any matrix A, $A^T A$ is symmetric and only has non-negative eigenvalues. *Hint:* Consider the quantity $\|A\vec{v}\|^2$. Remember that norms are positive-definite.

Answer

$A^T A$ is symmetric since $\left(A^T A\right)^T = A^T(A^T)^T = A^T A$ equals itself.

Let λ be an eigenvalue of $A^T A$ with corresponding eigenvector $\vec{v}$.

$$A^T A\vec{v} = \lambda\vec{v}$$

We left-multiply $\vec{v}^T$ on both sides.

$$\vec{v}^T A^T A\vec{v} = \vec{v}^T\lambda\vec{v}$$
$$\left(A\vec{v}\right)^T A\vec{v} = \lambda\vec{v}^T\vec{v}$$
$$\left\|A\vec{v}\right\|^2 = \lambda\|\vec{v}\|^2$$
$$\lambda = \frac{\left\|A\vec{v}\right\|^2}{\|\vec{v}\|^2} \geq 0$$

Note that $\|\vec{v}\| \neq 0$ because we assumed that $\vec{v}$ is an eigenvector corresponding to λ.

3 Outer Products

An **outer product** $\otimes$ is a function that takes two vectors and outputs a **matrix**. We define $\vec{x} \otimes \vec{y} = \vec{x}\vec{y}^T$.

a) Let $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$.

(i) Compute the outer-product $A = \vec{x}\vec{y}^T$.

Answer

$$\vec{x}\vec{y}^T = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \begin{bmatrix} 4 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & -1 \\ 12 & 6 & -3 \\ -8 & -4 & 2 \end{bmatrix}$$

(ii) What is the shape of the matrix A?

Answer

$\vec{x}\vec{y}^T$ is a 3×3 matrix.

(iii) What is the rank of A?

Answer

$\text{Rank}(A) = 1$.

b) Let $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

(i) Write B as an outer-product of two vectors $\vec{x}$ and $\vec{y}$.

Answer

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

(ii) What is the rank of B?

Answer

$\text{Rank}(B) = 1$.

c) Let $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

(i) Write C as a sum of outer-products: $\vec{x}\vec{y}^T + \vec{u}\vec{w}^T$.

Answer

$$C = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

(ii) What is the rank of C?

Answer

$\text{Rank}(C) = 2$.

d) Let $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(i) Write D as a sum of outer-products.

Answer

$$D = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

(ii) What is the rank of D?

Answer

$\text{Rank}(D) = 3$.