1 Scalar feedback control

Suppose that x has the following discrete-time dynamics:

$$x(t+1) = \lambda x(t) + bu(t), \quad x(0) = x_0 \tag{1}$$

a) Assuming that $x_0 = 1$ and $u = 0$, sketch $x(t)$ for a few time steps for $\lambda \in \{-1.1, -1, -0.5, 0.5, 1, 1.1\}$.

Answer

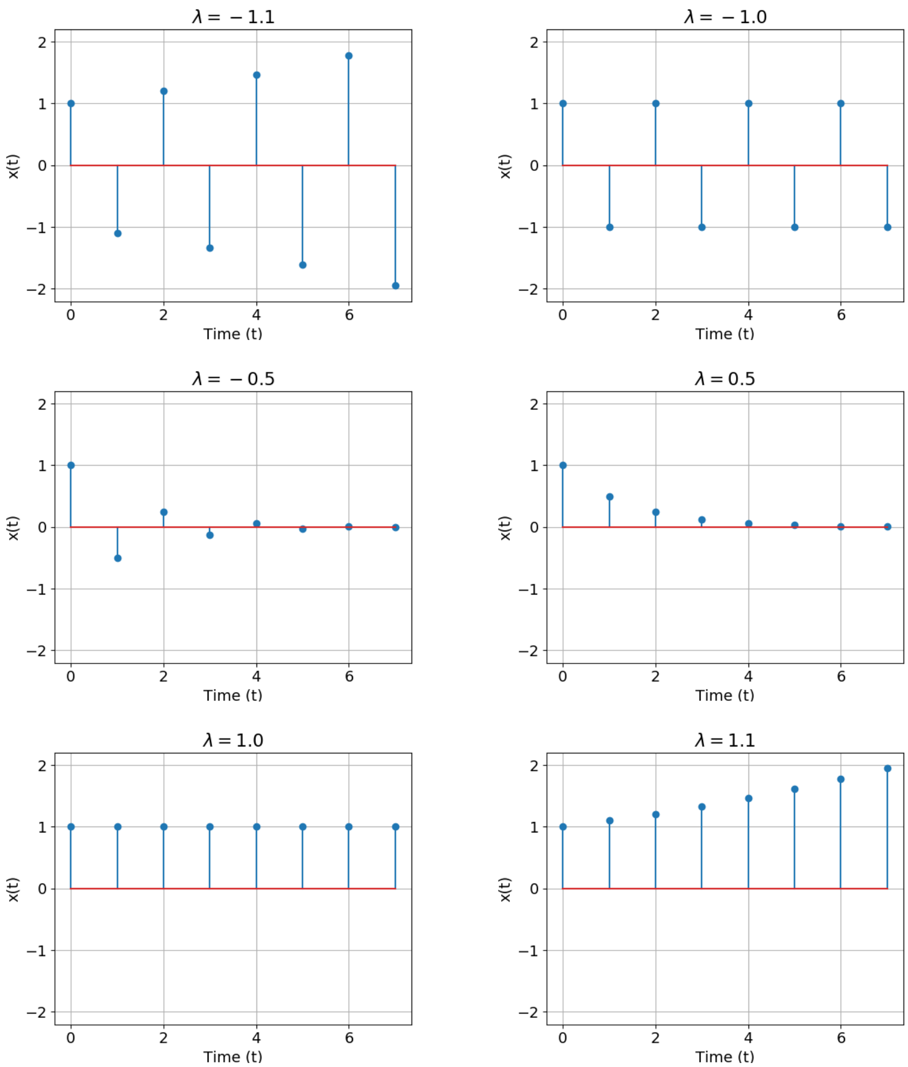

Figure 1: Response $x(t)$ for different values of λ.

b) What values of λ will result in convergence of x to its equilibrium? A scalar system having such a λ is called *stable*.

Answer

Let us look at the case when $u(t) = 0$. If $u(t) \neq 0$ and an equilibrium exists, a similar argument can be made. We know from our eigenvalue test that for this system to be stable, we need our eigenvalue to have a magnitude less than 1.

This results in $|\lambda| < 1$.

The case with $\lambda = 1$ is particularly interesting. For $u(t) = 0$, we have an equilibrium at $x = 0$. While in this scenario, $x(t)$ converges, it converges to a value of 1, which suspiciously is its initial condition. Think about what happens when we perturb this $x(t)$ from its initial value of $x(0) = 1$. Where does this perturbed system settle?

c) If $u(t) = u_0$ and the system is stable, what does x converge to? Sketch stable trajectories of x for $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.

Answer

Solve for states that statisfy $x(t+1) = x(t)$.

$$x = \lambda x + bu_0 \tag{2}$$

$$x = \frac{bu_0}{1-\lambda} \tag{3}$$

Notice that this equilibrium is approximately bu_0 if $\lambda \approx 0$, and that it grows without bound as $|\lambda| \to 1$.

d) If $x(t+1) = \lambda x(t) + bu(t)$ is unstable, describe feedback laws $u(t) = kx(t)$ that stabilize the equilibrium $x = 0$.

Answer

Substituting $u(t) = kx(t)$,

$$x(t+1) = \lambda x(t) + bkx(t) \tag{4}$$

$$= (\lambda + bk)\, x(t) \tag{5}$$

For stability of $x = 0$ we require

$$|\lambda + bk| < 1 \tag{6}$$

$$-1 < \lambda + bk < 1 \tag{7}$$

$$-1 - \lambda < bk < 1 - \lambda \tag{8}$$

Thus the stability criterion on k is

$$\begin{cases} -\frac{1+\lambda}{b} < k < \frac{1-\lambda}{b}, & b > 0 \\ \frac{1-\lambda}{b} < k < -\frac{1+\lambda}{b}, & b < 0 \end{cases} \tag{9}$$

e) Now, consider the continuous time system

$$\frac{d}{dt}x(t) = \lambda x(t) + bu(t) \tag{10}$$

Consider the case where this system is unstable ($\lambda \geq 0$). Design a feedback law $u(t) = kx(t)$ which stabilizes the equilibrium $x = 0$. You can assume that $b > 0$.

Answer

Using state feedback $u(t) = kx(t)$, we can rewrite our system as

$$\begin{aligned}\frac{d}{dt}x(t) &= \lambda x(t) + bkx(t) \\ &= (\lambda + bk)\, x(t)\end{aligned}$$

For this system to be stable, we need

$$\begin{aligned}\lambda + bk &< 0 \\ k &< -\frac{\lambda}{b}\end{aligned}$$

2 Eigenvalues Placement in Discrete Time

Consider the following linear discrete time system

$$\vec{x}(t+1) = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \tag{11}$$

a) Is this system controllable?

Answer

We calculate

$$C = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Observe that C matrix is full rank and hence our system is controllable.

b) Is the linear discrete time system stable?

Answer

We have to calculate the eigenvalues of matrix A. Thus,

$$\det(\lambda I - A) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = -2$$

Since the magnitudes of the eigenvalues λ_1 and λ_2 are greater than 1, the system is unstable.

c) Derive a state space representation of the resulting closed loop system using state feedback of the form $u(t) = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \vec{x}(t)$

Answer

The closed loop system using state feedback has the form

$$\begin{aligned} \vec{x}(t+1) &= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ &= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \left(\begin{bmatrix} k_1 & k_2 \end{bmatrix} \vec{x}(t) \right) \\ &= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right) \vec{x}(t) \end{aligned}$$

Thus, the closed loop system has the form

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} k_1 & 1+k_2 \\ 2 & -1 \end{bmatrix}}_{A_{\text{cl}}} \vec{x}(t)$$

d) Find the appropriate state feedback constants, k_1, k_2 in order the state space representation of the resulting closed loop system to place the eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$

Answer

$k_1 = 1, k_2 = -\frac{11}{8}$

Answer

From the previous part we have computed the closed loop system as

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} k_1 & 1+k_2 \\ 2 & -1 \end{bmatrix}}_{A_{\text{cl}}} \vec{x}(t)$$

Thus, finding the eigenvalues of the above system we have

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} k_1 - \lambda & 1+k_2 \\ 2 & -1-\lambda \end{bmatrix} = \lambda^2 + (1-k_1)\lambda + (-k_1 - 2k_2 - 2)$$

We want to place the eigenvalue at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$. This means that we should choose the gains k_1 and k_2 so that the characteristic equation is

$$0 = (\lambda - \frac{1}{2})(\lambda + \frac{1}{2}) = \lambda^2 - \frac{1}{4}.$$

Thus we should choose k_1 and k_2 satisfying the system of equations

$$\begin{aligned} 0 &= 1 - k_1 \\ -\frac{1}{4} &= -k_1 - 2k_2 - 2 \end{aligned}.$$

This system has solution $k_1 = 1, k_2 = -\frac{11}{8}$.

e) Suppose that instead of $\begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$ in (11), we had $\begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$ as the way that the discrete-time control acted on the system. Is this system controllable from $u(t)$?

Answer

We calculate

$$C = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Observe that C matrix is not full rank and hence our system is not controllable.

f) For the part above, suppose we used $[k_1, k_2]$ to try and control the system. What would the eigenvalues be? Can you move all the eigenvalues to where you want? Give an intuitive explanation of what is going on.

Answer

$$\vec{x}(t+1) = \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} k_1 & k_2 \end{bmatrix}\right)\vec{x}(t) \tag{12}$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ k_1 & k_2 \end{bmatrix}\right)\vec{x}(t) \tag{13}$$

$$\tag{14}$$

Finding the eigenvalues λ:

$$\det\left(\begin{bmatrix} k_1 - \lambda & k_2 + 1 \\ k_1 + 2 & k_2 - 1 - \lambda \end{bmatrix}\right) = 0 \tag{15}$$

$$= (k_1 - \lambda)(k_2 - 1 - \lambda) - (k_1 + 2)(k_2 + 1) \tag{16}$$

$$= k_1(k_2 - 1) - k_1\lambda - \lambda(k_2 - 1) + \lambda^2 - (k_1k_2 + k_1 + 2k_2 + 2) \tag{17}$$

$$= k_1k_2 - k_1 - k_1\lambda - \lambda k_2 + \lambda + \lambda^2 - k_1k_2 - k_1 - 2k_2 - 2 \tag{18}$$

$$= \lambda^2 + (1 - k_1 - k_2)\lambda - 2(1 + k_1 + k_2) \tag{19}$$

We can now use the quadratic formula to find the roots of this polynomial. These roots are

$$\lambda = \frac{-(1 - k_1 - k_2) \pm \sqrt{(1 - k_1 - k_2)^2 - 4(-2(1 + k_1 + k_2))}}{2} \tag{20}$$

$$= \frac{-(1 - k_1 - k_2) \pm \sqrt{1 + k_1^2 + k_2^2 - 2k_1 - 2k_2 + 2k_1k_2 + 8(1 + k_1 + k_2)}}{2} \tag{21}$$

$$= \frac{-(1 - k_1 - k_2) \pm \sqrt{9 + k_1^2 + k_2^2 + 6k_1 + 6k_2 + 2k_1k_2}}{2} \tag{22}$$

$$= \frac{-(1 - k_1 - k_2) \pm \sqrt{(3 + k_1 + k_2)^2}}{2} \tag{23}$$

$$= \frac{-1 + k_1 + k_2 \pm (3 + k_1 + k_2)}{2} \tag{24}$$

$$\lambda \in \{-2, 1 + k_1 + k_2\} \tag{25}$$

We can see that the eigenvalue at $\lambda = -2$ cannot be moved, so we cannot arbitrarily change our eigenvalues with this control input.