1 Stability

A system is *stable* if $\vec{x}(t)$ is bounded for any initial condition $\vec{x}(0)$ and any bounded input $u(t)$. A system is *unstable* if there is an $\vec{x}(0)$ and some bounded input $u(t)$ for which $|\vec{x}(t)| \to \infty$ as $t \to \infty$.

Discrete time systems

A discrete time system is of the form:

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t)$$

Let λ be any particular eigenvalue of A. This system is stable if for all λ, $|\lambda| < 1$. This system is unstable if there exists an eigenvalue λ, $|\lambda| \geq 1$.

Continuous time systems

A continuous time system is of the form:

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}(t) = A\vec{x}(t) + B\vec{u}(t)$$

Let λ be any particular eigenvalue of A. This system is stable if for all λ, $\mathrm{Re}\{\lambda\} < 0$. This system is unstable if there exists an eigenvalue λ, $\mathrm{Re}\{\lambda\} \geq 0$.

2 Controllability

We are given a discrete time state space system, where $\vec{x}$ is our state vector, A is the state space model, B is the input matrix, and $\vec{u}$ is the control input.

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t)$$

We want to know if this system is "controllable"; if given set of inputs, we can get the system from any initial state to any final state. This has an important physical meaning; if a physical system is controllable, that means that we can get anywhere in the state space. If a robot is controllable, it is able to travel anywhere in the system it is living in (given enough control inputs).

Constructing the Controllability Matrix

To figure out if a system is controllable, we can simplify the problem. If we want to reach any final state from any initial state, we can consider the initial state as the origin and the final state as any arbitrary point in the state space. A system is controllable if we start off at the initial state $\vec{x}(0) = \vec{0}$ at time $t = 0$, and after some set of control inputs $\vec{u}(t)$, we can reach an arbitrary final state $\vec{x}_0$. Let's start the system off at $\vec{x}(0)$ and see how the system evolves with each time step.

$$\vec{x}(1) = A\vec{x}(0) + B\vec{u}(0) = A\vec{0} + B\vec{u}(0) = B\vec{u}(0)$$

This shows us that we can go anywhere spanned by B in the first time step. Using our input vector $\vec{u}$, we can push the system anywhere the matrix B lets us go. Now consider the next time step.

$$\begin{aligned}\vec{x}(2) &= A\vec{x}(1) + B\vec{u}(1) \\ &= AB\vec{u}(0) + B\vec{u}(1)\end{aligned}$$

Similarly, at this time step, we can go anywhere spanned by $\begin{bmatrix} AB & B \end{bmatrix}$. Every time step adds another degree of freedom to the system.

If we go another time step, $\vec{x}(3)$, we get the following:

$$\begin{aligned}\vec{x}(3) &= A\vec{x}(2) + B\vec{u}(2)\\ &= A^2B\vec{u}(0) + AB\vec{u}(1) + B\vec{u}(2)\end{aligned}$$

After k time steps, we get the following:

$$\begin{aligned}\vec{x}(k) &= A\vec{x}(k-1) + B\vec{u}(k-1)\\ &= A^{k-1}B\vec{u}(0) + A^{k-2}B\vec{u}(1) + A^{k-3}B\vec{u}(2) + \cdots + AB\vec{u}(k-2) + B\vec{u}(k-1)\end{aligned}$$

After 1 time step, we can go anywhere in the set of vectors spanned by B, after 2 time steps, we can go anywhere spanned by $\begin{bmatrix}AB & B\end{bmatrix}$, and after k time steps, we can go anywhere spanned by the columns of the matrix C defined below. This is called the "controllability" matrix.

$$C = \begin{bmatrix}A^{k-1}B & A^{k-2}B & \cdots & A^2B & AB & B\end{bmatrix}$$

If this matrix is of rank n (the dimension of our state space), then our system is controllable. It means that our control system is a surjection from the domain of control inputs to the state space. But what if these aren't enough steps and the system can be controlled only in $k+1$ steps? What is the maximal number of steps we need to take to have a long sequence of control inputs that $\left\{A^{k-1}B, A^{k-2}B, \cdots, AB, B\right\}$ spans the state space?

Cayley-Hamilton Theorem

These questions are answered by the Cayley-Hamilton theorem. The Cayley-Hamilton theorem says that higher order powers of A can be expressed as a linear combination of lower order matrix powers of A. Specifically if A is an $n \times n$, matrix, the highest order unique power of A is A^{n-1}. Thus, if we keep applying control inputs past n time steps, our control inputs will be a linear combination of the previous control inputs and cannot increase the rank of the controllability matrix.

Definition of Controllability

It works for both discrete time systems and continuous time systems [1] that we can determine the controllability of the system by testing the rank of controllability matrix C up to only the $A^{n-1}B$ term. Because higher order terms (any higher than $n-1$) of A matrix can be written as linear combination of power of A with A^{n-1} being the highest order term. And thus A^kB with $k > n-1$ can always be written as linear combination of $\{A^{n-1}B, A^{n-2}B, \cdots, A^2B, AB, B\}$.

$$\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t) \quad \text{or} \quad \vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t)$$
$$C = \begin{bmatrix}A^{n-1}B & A^{n-2}B & \cdots & A^2B & AB & B\end{bmatrix}$$

Given a continuous or discrete time system shown above with state vector $\vec{x}$ of dimension n, the system is controllable if its controllability matrix C is of rank n.

If a discrete time system is controllable, then given a starting position $\vec{x}(0) = \vec{0}$, it takes a maximum of n control inputs over n time steps for the system to reach any final state $\vec{x}_0$. It also holds that it takes a maximum of n control inputs over n time steps for the system to reach the origin from any starting position $\vec{x}(0)$.

If a continuous time system is controllable, then given a starting position $\vec{x}(0) = \vec{0}$, there exists some $u(t)$ that drives the state $\vec{x}(t')$ to any final state for any $t' > 0$. It also holds that there exists some $u(t)$ that drives the state $\vec{x}(t')$ to origin for any $t' > 0$ from any starting position $\vec{x}(0)$.

[1] The proof would be more involved but it bases on the same idea.

3 Deadbeat Control

Consider the discrete-time system

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

a) Is this system stable if $u(t) = 0$ for all t?

Answer

Observe that A is already in the upper diagonal form and is *not diagonalizable*. The eigenvalues $\lambda_1 = \lambda_2 \geq 1$. Thus we conclude that the system is unstable.

To see this in practice, we can also derive the trajectory with the initial state (starting position) being $\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We note that $\vec{x}(t) = \begin{bmatrix} -t \\ 1 \end{bmatrix}$ and $|x_1(t)| \longrightarrow \infty$ as $t \longrightarrow \infty$.

b) Is this system controllable?

Answer

We compute the controllability matrix C:

$$C = \begin{bmatrix} AB & B \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

This matrix has a rank of 2, so the system is controllable.

c) For which initial states $\vec{x}(0)$ is there a control that will bring the state to zero in a single time step?

Answer

To find the initial states that can be brought to zero in a single step, we solve:

$$\begin{aligned}
\begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(0) \\
&= \begin{bmatrix} x_1(0) - x_2(0) \\ x_2(0) + u(0) \end{bmatrix} \\
\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} &= \begin{bmatrix} x_1(0) - x_2(0) \\ x_2(0) + u(0) \end{bmatrix} \\
\implies 0 &= x_1(0) - x_2(0).
\end{aligned}$$

Therefore, there is a one-dimensional subspace $\{x_1(0) - x_2(0) = 0\}$ of initial states that can be brought to zero in one step.

d) For which initial states $\vec{x}(0)$ is there a control that will bring the state to zero in two time steps?

Answer

To find the initial states that can be brought to zero in two steps, we solve:

$$\begin{aligned}
\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(1) \\
&= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) - x_2(0) \\ x_2(0) + u(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(1) \\
&= \begin{bmatrix} x_1(0) - 2x_2(0) - u(0) \\ x_2(0) + u(0) + u(1) \end{bmatrix} \\
\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} &= \begin{bmatrix} x_1(0) - 2x_2(0) - u(0) \\ x_2(0) + u(0) + u(1) \end{bmatrix}
\end{aligned}$$

Therefore, any initial state can be brought to zero in two steps using an appropriate choice of inputs $u(0) = x_1(0) - 2x_2(0)$ and $u(1) = -x_2(0) - u(0) = x_2(0) - x_1(0)$.

Another way to view this is to note that we want to find $u(0)$ and $u(1)$ such that the following equations hold.

$$\begin{aligned}
\begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= A^2 \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + ABu(0) + Bu(1) \\
\implies -A^2 \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} &= \begin{bmatrix} AB & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \end{bmatrix}
\end{aligned}$$

Because the span of $\begin{bmatrix} AB & B \end{bmatrix}$ is $\mathbb{R}^2$ (verified by the fact that the controllability matrix is rank 2), for any $\vec{x}$ there exists a vector $\begin{bmatrix} u(0) \\ u(1) \end{bmatrix}$ such that $\vec{x} = \begin{bmatrix} AB & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \end{bmatrix}$. Thus no matter what initial state $x_1(0), x_2(0)$ is, there exists $u(0)$ and $u(1)$ that drives $x_1(2)$ and $x_2(2)$ to 0.

4 Cayley and Hamilton

Cayley is trying to control the continuous-time linear system

$$\frac{\mathrm{d}}{\mathrm{d}t} \vec{x}(t) = A\vec{x}(t) + Bu(t) = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

a) Is this system stable?

Answer

The eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = -2$. Because there exists an eigenvalue λ_1 that $\mathrm{Re}\{\lambda_1\} = 4 \geq 0$, the system is not stable.

b) Is this system controllable?

Answer

We compute the controllability matrix C:

$$C = \begin{bmatrix} AB & B \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}$$

This matrix has rank 1, so the system is not controllable.

c) Cayley has been trying to find some k, such that the matrix

$$C_k = \begin{bmatrix} A^k B & \cdots & A^2 B & AB & B \end{bmatrix}$$

has rank 2 but still hasn't found one. Confirm that for $k = 3$, this matrix still has rank 1.

Answer

$$C_3 = \begin{bmatrix} A^3 B & A^2 B & AB & B \end{bmatrix} = \begin{bmatrix} -8 & 4 & -2 & 1 \\ -8 & 4 & -2 & 1 \end{bmatrix}$$

This matrix still has rank 1.

d) Cayley's friend Hamilton remembers hearing somewhere that for any $n \times n$ matrix A, the matrix A^n can always be written as a linear combination of $A^{n-1}, A^{n-2}, \ldots, A$, and I. [2] Check that this is true for the A matrix of Cayley's system. Check that $A^2 = \alpha A + \beta I$ with appropriate α and β.

Answer

We want to find some coefficients α and β, such that

$$\begin{bmatrix} 10 & -6 \\ -6 & 10 \end{bmatrix} = A^2 = \alpha A + \beta I = \begin{bmatrix} \beta + \alpha & -3\alpha \\ -3\alpha & \beta + \alpha \end{bmatrix}.$$

If we choose $\alpha = 2$ and $\beta = 8$, we can make this equation hold.

e) Will Cayley ever find some k to make

$$C_k = \begin{bmatrix} A^k B & \ldots & A^2 B & AB & B \end{bmatrix}$$

have rank 2?

Answer

No. Since A^2 is just a linear combination of A and I and $A^2 B$ is just a linear combination of AB and B, repeatedly exponentiating A will never get him any more linearly independent matrices: for $k > 1$, the span of $\{A^k B, A^{k-1} B, \cdots, AB, B\}$ equals the span of $\{AB, B\}$. Therefore, for any k, he will never be able to make C_k have rank 2.

[2] Hamilton is right about this. It follows from the Cayley-Hamilton Theorem, which says that any $n \times n$ matrix always satisfies its characteristic equation. Therefore, the characteristic equation $\lambda^2 - 2\lambda - 8$ we derived above implies that $A^2 - 2A - 8 = 0$. You'll learn more about this theorem if you take the advanced control course EE 221A.