1 Conditions for Equilibria

Continuous-Time Systems

Let us take a closer look at the conditions for a linear system represented by the differential equation

\[
\frac{d}{dt} \dot{x}(t) = A\dot{x}(t) + B\ddot{u}(t) \quad (1)
\]

From the get-go we see that \((\dot{x}^*, \ddot{u}^*) = (\vec{0}, \vec{0})\) must be an equilibrium point. This is since the system is at rest. Now if we put in a constant input \(\ddot{u}^*\) then to solve for equilibria, we get the following system of equations

\[
A\dot{x} + B\ddot{u}^* = \vec{0} \quad (2)
\]

To solve for the states \(\dot{x}\) in which the system would be in equilibrium, our analysis boils down to whether the square matrix \(A\) is invertible.\(^1\)

a) If \(A\) is invertible, then there is a unique equilibrium point \(\dot{x}^* = -A^{-1}B\ddot{u}^*\).

b) If \(A\) is non-invertible, depending on the range of \(A\), we have two scenarios.

- If \(B\ddot{u} \in \text{Col}(A)\) then we will have infinitely many equilibrium points.
- If \(B\ddot{u} \notin \text{Col}(A)\) then the system has no solution and we will have no equilibrium points.

Discrete-Time Systems

Now let’s take a look at the discrete-time system

\[
\dot{x}(t + 1) = A\dot{x}(t) + B\ddot{u}(t) \quad (3)
\]

Again we see that \((\vec{0}, \vec{0})\) is an equilibrium point but notice that the conditions for equilibria are different for discrete-time systems. A system is in equilibrium if it is not changing. In otherwords, this means that \(\dot{x}^*(t + 1) = \dot{x}^*(t)\) therefore, for a constant input \(\ddot{u}^*\) we get the following system of equations

\[
\dot{x} = A\dot{x} + B\ddot{u}^* \quad \implies (I - A)\dot{x} = B\ddot{u}^* \quad (4)
\]

The conditions for equilibria now depend on the matrix \(I - A\) being invertible instead of the matrix \(A\).

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\(^1\)This should be review from 16A/54, but we restate it here since it isn’t quite obvious when \(A\) is singular or non-invertible. Normally a singular matrix has infinite solutions but take the system \(A\vec{x} = \vec{b}\) with \(A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\) and \(\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\). This leads to a contradiction that \(x_1 = 0 \neq 1\).
2 Stability

Continuous time systems

A continuous time system is of the form:

\[
\frac{d\vec{x}(t)}{dt} = A\vec{x}(t) + B\vec{u}(t)
\]

This system is stable if \(\text{Re}\{\lambda_i\} < 0\) for all \(\lambda_i\)'s are the eigenvalues of \(A\). If we plot all \(\lambda_i\) for \(A\) on the complex plane, if all \(\lambda_i\) lie to the left of \(\text{Re}\{\lambda_i\} = 0\), then the system is stable.

If \(\text{Re}\{\lambda_i\} \geq 0\), the system is unstable in the context of BIBO stability.
Discrete time systems

A discrete time system is of the form:

\[ \tilde{x}(t + 1) = A\tilde{x}(t) + B\tilde{u}(t) \]

This system is stable if \(|\lambda_i| < 1\) for all \(\lambda_i\), where \(\lambda_i\)'s are the eigenvalues of \(A\). If we plot all \(\lambda_i\) for \(A\) on the complex plane, if all \(\lambda_i\) lie within (not on) the unit circle, then the system is stable.

If \(|\lambda| \geq 1\), we say the system is unstable in the context of Bounded-Input Bounded-Output (BIBO) stability.
3 Jacobian Warm-Up

Consider the following function $f : \mathbb{R}^2 \mapsto \mathbb{R}^3$

$$f(x_1, x_2) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \\ f_3(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_1^2 - e^{x_2^2} \\ x_1^2 + \sin(x_1)x_2^2 \\ \log(1 + x_1^2) \end{bmatrix}$$

Calculate its Jacobian.

Answer

$$\frac{df}{dx} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 & -2x_2e^{x_2^2} \\ 2x_1 + \cos(x_1)x_2^2 & 2\sin(x_1)x_2 \\ \frac{2x_1}{1 + x_1^2} & 0 \end{bmatrix}$$
4 Linearization

Consider a mass attached to two springs:

We assume that each spring is linear with spring constant $k$ and resting length $X_0$. We want to build a state space model that describes how the displacement $y$ of the mass from the spring base evolves. The differential equation modeling this system is $\frac{d^2y}{dt^2} = -\frac{2k}{m}(y - X_0 \frac{y}{\sqrt{y^2 + a^2}}).

a) Write this model in state space form $\dot{x} = f(x)$.

Answer

We introduce states $x_1 = y$ and $x_2 = \dot{y}$. Writing the model in state space form gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{2k}{m} \left( x_1 - X_0 \frac{x_1}{\sqrt{x_1^2 + a^2}} \right) \end{bmatrix}.$$

b) Find the equilibrium of the state-space model. You can assume $X_0 < a$.

Answer

We find the equilibrium by solving $0 = \dot{x} = f(x)$:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{2k}{m} \left( x_1 - X_0 \frac{x_1}{\sqrt{x_1^2 + a^2}} \right) \end{bmatrix}.$$

The unique solution is the equilibrium at $(x_1, x_2) = (0, 0)$.

c) Linearize your model about the equilibrium.

Answer

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{x=(0,0)} \bigg|_{x=(0,0)} = \begin{bmatrix} 0 & \frac{-2k}{m} \left( 1 - X_0 \frac{a^2}{(x_1^2 + a^2)^{3/2}} \right) \\ 0 & \frac{-2k}{m} \left( 1 - X_0 \frac{a}{a} \right) \end{bmatrix} \bigg|_{x=(0,0)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$
So the linearized system is
\[ \dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{2k}{m} \left(1 - \frac{X_0}{a}\right) & 0 \end{bmatrix} x. \]

\[ \begin{bmatrix} 1 - \frac{X_0}{a} \\ 0 \end{bmatrix} < 0. \]
\[ \begin{bmatrix} 1 - \frac{X_0}{a} \\ 0 \end{bmatrix} > 0. \]
\[ \lambda = \pm \sqrt{\frac{2k}{m} \left(1 - \frac{X_0}{a}\right)} j. \]

Since the linearized system has purely imaginary eigenvalues that are not repeated, their real parts are zero. Therefore the equilibrium is unstable.

**d) Compute the eigenvalues of your linearized model. Is this equilibrium stable?**

**Answer**
To compute the eigenvalues, we solve
\[ 0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} -\frac{2k}{m} \left(1 - \frac{X_0}{a}\right) & 1 \\ -\lambda & -\lambda \end{bmatrix}\right) = \lambda^2 + \frac{2k}{m} \left(1 - \frac{X_0}{a}\right). \]

Since \(X_0 < a\), this means that \(1 - \frac{X_0}{a}\) > 0. So we have a pair of imaginary eigenvalues
\[ \lambda = \pm \sqrt{\frac{2k}{m} \left(1 - \frac{X_0}{a}\right)} j. \]
5 Stability in discrete time system

Determine which values of $\alpha$ and $\beta$ will make the following discrete-time state space models stable. Assume, $\alpha$ and $\beta$ are real numbers and $b \neq 0$.

a) $x(t + 1) = \alpha x(t) + bu(t)$

**Answer**

$|\alpha| < 1$

b) $\tilde{x}(t + 1) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \tilde{x}(t) + b\tilde{u}(t)$

**Answer**

The eigenvalues of this system are:

$\lambda = \alpha \pm j\beta$

$|\lambda| = \sqrt{\alpha^2 + \beta^2}$

For this system to be stable, $|\lambda| < 1$, so

$\alpha^2 + \beta^2 < 1$

c) $\tilde{x}(t + 1) = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \tilde{x}(t) + b\tilde{u}(t)$

**Answer**

The eigenvalues of this system are

$\lambda = 1, 1$

This means that regardless of $\alpha$, this system is always unstable since $|\lambda| \geq 1$. 