1 State Space Models

There are many kinds of **dynamical systems** we might want to study or control. Some examples are an airplane's flight, the air inside a building, or network traffic on the internet. We can develop controllers for these systems to regulate particular quantities that we care about, like an autopilot to level an airplane's flight, a thermostat to keep a building at a comfortable temperature, or internet congestion control to manage data rates. Other dynamical systems and controllers can be found in nature, like the biochemical systems that regulate conditions inside a living cell.

When we want to study or control a dynamical system, our first step is usually to write out equations that describe its physics. These equations are called a **model**, and they predict what a system will do over time. We will study systems that change continuously in time like electrical circuits, and systems that evolve in discrete time steps, like the yearly number of professors in EECS.

State variables are a set of variables that fully represent the state of a dynamical system at a given time, like capacitor voltages and inductor currents in electrical circuits. In a mechanical system, they could be the positions and velocities of masses. The state variables can be written together in a **state vector** $\vec{x}(t) \in \mathbb{R}^n$ where n is the number of state variables that describe the system.

2 Continuous Systems

For a **continuous-time system**, the dynamics can be described by n first-order differential equations:

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t))$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a function of the state vector that returns the time derivative of the state vector (which is an n-dimensional vector containing the time derivative of each state variable).

A system with m input signals can be described as:

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t), \vec{u}(t))$$

where $\vec{u}(t) \in \mathbb{R}^m$ is a **control input** with which we can vary to influence the system.

We can expand out this vector dynamics equation:

$$\begin{bmatrix} \frac{d}{dt}x_1(t) \\ \vdots \\ \frac{d}{dt}x_n(t) \end{bmatrix} = \begin{bmatrix} f_1(\vec{x}(t), \vec{u}(t)) \\ \vdots \\ f_n(\vec{x}(t), \vec{u}(t)) \end{bmatrix},$$

where $f_i(\vec{x}, \vec{u}(t))$ returns the time derivative of the ith state variable. A continuous-time system is **linear** if it can be expressed in the form $\frac{d}{dt}\vec{x}(t) = \mathbf{A}\vec{x}(t) + \mathbf{B}\vec{u}(t)$:

3 Discrete Time Systems

For a **discrete-time system**, the dynamics can be described by n difference equations:

$$\vec{x}[t+1] = f(\vec{x}[t], \vec{u}[t]),$$

where $\vec{x}[t+1]$ is the new state vector at the next time step.

As in the continuous case, a linear discrete-time system's dynamics can be written as:

$$\vec{x}[t+1] = \mathbf{A}\vec{x}[t] + \mathbf{B}\vec{u}[t]$$

4 Dirty Dishes

I am a trip planner who lodges travellers at Bob's Bed and Breakfast. At the beginning of each day, Bob will do half of the dirty dishes in the sink. During the day, each of his guests will use 4 pounds of dishes minus an eighth pound of dishes for each pound of dishes already in the sink at the beginning of the day (as Bob's kitchen gets too messy).

a) Define a state vector for Bob's kitchen sink system. Also, what are the inputs? Lastly, write out the state space model using your state vector and inputs.

Answer

The dishes in the sink are the state variable x. The number of guests are the input u.

$$x[t+1] = \frac{1}{2}x[t] + \left(4 - \frac{1}{8}x[t]\right)u[t]$$

b) Explain why Bob's kitchen is not a linear system.

Answer

It is not a linear system since the state variable is multiplied by the input.

c) On Wednesday morning (before Bob gets up), there are 4 pounds of dishes in the sink. On Wednesday, Bob has 4 guests, and on Thursday, he has 5 guests. How many pounds of dishes are in the sink after Thursday?

Answer

$$x[1] = \frac{1}{2}(4) + \left(4 - \frac{1}{8}(4)\right)(4) = 16$$
$$x[2] = \frac{1}{2}(16) + \left(4 - \frac{1}{8}(16)\right)(5) = 18$$

d) I am a very eccentric trip planner and I want Bob to have exactly 12 pounds of dishes in his sink. He has 24 pounds of dishes in his sink. How many guests should I lodge at Bob's Bed and Breakfast today? How many guests should I lodge tomorrow?

Answer

$$12 = \frac{1}{2}(24) + \left(4 - \frac{1}{8}(24)\right)u[0]$$
$$u[0] = 0$$

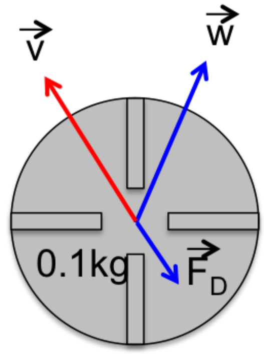
$$12 = \frac{1}{2}(12) + \left(4 - \frac{1}{8}(12)\right)u[1]$$
$$u[1] = \frac{12}{5}$$

5 Remote Control Hovercraft

Taejin has a toy hovercraft that he can drive around on the ground. It weighs $0.1\,\text{kg}$. His remote control has two levers: one sets the thrust in the x-direction, w_x, measured in Newtons, and the other sets the thrust in the y-direction, w_y, measured in Newtons. The hovercraft experiences a drag force:

$$\vec{F} = -D\vec{v},$$

where $\vec{F}$ is the drag force vector in Newtons, $\vec{v}$ is the hovercraft velocity vector in $\frac{\text{m}}{\text{s}}$, and D is the coefficient $0.05\,\frac{\text{N s}}{\text{m}}$.



a) If we are interested in both the position and velocity of our hovercraft, define the appropriate state variables and inputs for the hovercraft system.

Answer

The state vector should include the x and y positions and the x and y velocities. It can be arranged as:

$$\begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix}$$

The inputs are thursts w_x and w_y. The input vector could be arranged as:

$$\begin{bmatrix} w_x \\ w_y \end{bmatrix}$$

b) Write out the state space model using the state variables and inputs you identified.

Hint: Newton's Second Law states that $F_{net} = ma$. Also, remember that $a(t) = v'(t)$ and $v(t) = x'(t)$.

Answer

First, let's write out the force balances in the x and y directions using Newton's second law: $F = ma$.

$$\begin{aligned} ma_x &= w_x - Dv_x \\ a_x &= \frac{1}{m}w_x - \frac{D}{m}v_x \\ ma_y &= w_y - Dv_y \\ a_y &= \frac{1}{m}w_y - \frac{D}{m}v_y \end{aligned}$$

Since $\vec{a} = \frac{d}{dt}\vec{v}$, we can now write out our equations:

$$\frac{d}{dt}x = v_x$$
$$\frac{d}{dt}y = v_y$$
$$\frac{d}{dt}v_x = \frac{1}{m}w_x - \frac{D}{m}v_x$$
$$\frac{d}{dt}v_y = \frac{1}{m}w_y - \frac{D}{m}v_y$$

Substituting in for D and m, we get:

$$\frac{d}{dt}x = v_x$$
$$\frac{d}{dt}y = v_y$$
$$\frac{d}{dt}v_x = 10w_x - 0.5v_x$$
$$\frac{d}{dt}v_y = 10w_y - 0.5v_y$$

c) Is this system linear? If it is, write it in the form $\frac{d}{dt}\vec{x} = \mathbf{A}\vec{x} + \mathbf{B}\vec{u}$. If it isn't, explain why not.

Answer

This system is linear. Using the state vector from the first part, we can write the dynamics as $\frac{d}{dt}\vec{x} = \mathbf{A}\vec{x} + \mathbf{B}\vec{u}$ where:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & -0.5 \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 10 & 0 \\ 0 & 10 \end{bmatrix}$$

d) Taejin places the hovercraft at $(1, 0)$. At $t = 0$, Ramsey kicks the hovercraft, so that it moves at $2\,\frac{\text{m}}{\text{s}}$ in the x direction, and Taejin doesn't touch the remote control. What will the position and velocity of the hovercraft be at $t = 10$?

Hint: Try to solve for the velocity first, then use the fact that $x'(t) = v(t)$ to solve for position.

Answer

Since there is no input from the remote control, all inputs are zero. There is no force on the hovercraft in the y-direction so $y = 0$ and $v_y = 0$.

Now looking at the x dynamics, we know from our system that $\frac{d}{dt}v_x = -0.5v_x$ and $\frac{d}{dt}x = v_x$. Since $v_x(0) = 2$ and the dynamics are a homogeneous first-order differential equation, we can say that

$$v_x(t) = 2e^{-0.5t}$$

Lastly, to solve for $x(t)$, we can take $\frac{d}{dt}x = v_x$ and integrate both sides from $t = 0$ to $t = 10$

$$\int_0^t x'(t)dt = \int_0^t v_x(t)dt$$

$$x(t) - x(0) = \int_0^t 2e^{-0.5t}dt$$

$$= -4e^{-0.5t} + 4$$

Plugging in the initial condition $x(0) = 1$, we see that

$$x(t) = 5 - 4e^{-0.5t}$$

This means at $t = 10$, the hovercraft is at $x = 5 - 4e^{-5}$ and is moving at speed $v_x = 2e^{-5}$. Note that the hovercraft is very close to $x = 5\,\mathrm{m}$ and v_x is close to $0\,\frac{\mathrm{m}}{\mathrm{s}}$.