Rational Transfer Functions

When we write the transfer function of an arbitrary circuit, it always takes the following form. This is called a "rational transfer function." We also like to factor the numerator and denominator, so that they become easier to work with and plot:

$$H(\omega) = K \frac{(j\omega)^{N_{z0}} \left(1 + j\frac{\omega}{\omega_{z1}}\right)\left(1 + j\frac{\omega}{\omega_{z2}}\right)\cdots\left(1 + j\frac{\omega}{\omega_{zn}}\right)}{(j\omega)^{N_{p0}} \left(1 + j\frac{\omega}{\omega_{p1}}\right)\left(1 + j\frac{\omega}{\omega_{p2}}\right)\cdots\left(1 + j\frac{\omega}{\omega_{pm}}\right)} \tag{1}$$

Here, we define the constants ω_z as "zeros" and ω_p as "poles."

Bode Plots

Bode plots provide us with a simple and easy tool to plot these transfer functions by hand. Always remember that Bode plots are an approximation; if you want the precisely correct plots, you need to use numerical methods (like solving using MATLAB or iPython).

When we make Bode plots, we plot the frequency and magnitude on a logarithmic scale, and the angle in either degrees or radians. We use the logarithmic scale because it allows us to break up complex transfer functions into its constituent components.

For two transfer functions $H_1(\omega)$ and $H_2(\omega)$, if $H(\omega) = H_1(\omega) \cdot H_2(\omega)$,

$$\log\left|H(\omega)\right| = \log\left|H_1(\omega) \cdot H_2(\omega)\right| = \log\left|H_1(\omega)\right| + \log\left|H_2(\omega)\right| \tag{2}$$

$$\angle H(\omega) = \angle(H_1(\omega) \cdot H_2(\omega)) = \angle H_1(\omega) + \angle H_2(\omega) \tag{3}$$

Decibels (Old notation you will find scattered over internet)

We define the decibel as the following:

$$20\log_{10}(|H(\omega)|) = |H(\omega)|\ [\text{dB}]$$

This means that $20\,\text{dB}$ per decade is equivalent to one order of magnitude. A decade is defined as a change in ω by a factor of 10. For example ω going from 1 to 10 is a decade.

NOTE: We won't be using dB when plotting, but understanding the conversion to dB will help when reading the Bode plot sheet on the next page.

Algorithm

Given a frequency response $H(\omega)$, you can follow the following steps to sketch the magnitude plot:

a) Break $H(\omega)$ into a product of poles and zeros and put it in "rational transfer function" form as described in Equation 1.

b) Sketch the Bode plot for the poles and zeros in an increasing order. Start with the smallest pole/zero (this could be a pole or zero at origin). If you encounter a zero, the slope of you magnitude plot increases by the order of the zero. For example, if you encounter a zero of order 2 at $\omega = 100$, the slope increases by 2 from that frequency. On the other hand, if you encounter a pole, the slope decreases by the order of the pole.

c) Add the resulting plots to get the final Bode plot.

We will not focus on plotting phase plots. However, the algorithm is similar to the one described for magnitude plots here. An order 1 zero, for example, results in a slope increase of $\frac{\pi}{4}$ rad/decade in the phase of $H(\omega)$. These are also illustrated in Figure 1

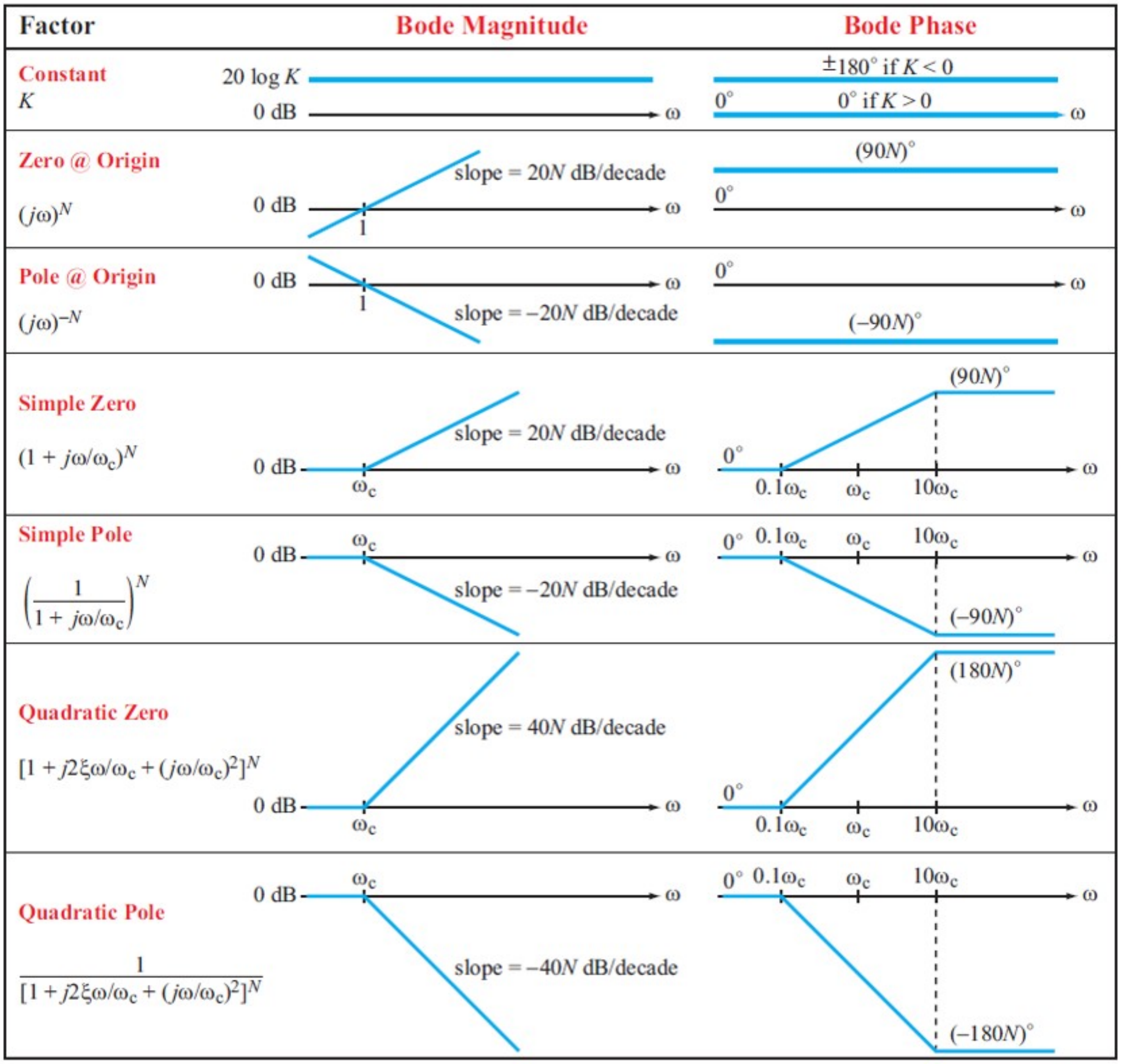


Figure 1: Reference for sketching Bode plots for transfer functions.

1 Log-Log plots

In this problem we will explore some of the ideas behind $\log - \log$ plots. Consider functions

$$f(x) = 100x^2\,,$$
$$g(x) = 10(1 + x^{10})$$

We will combine these functions in different ways to create a new function that we will call h(x). We will then analyze this newly created function. In particular, we are interested in the relationship between $\log_{10}(x)$ and $\log_{10} h(x)$ Let us assign a new name for $\log_{10}(x)$. From here, on we will call it $z \equiv \log_{10}(x)$.

Note that $\log_{10}(x)$ is well defined for $x > 0$ and we will restrict ourselves to x in that domain. The functions that we are looking at, are also strictly positive for $x > 0$.

a) $h(x) = f(x)$

Find a relationship between z and $H_1(z) \equiv \log_{10} h(x)$. What does this relationship resemble?

Answer

Let us start with taking a logarithm to base 10 for the function $h(x)$ given to us. We have

$$\begin{aligned}\log_{10} h(x) &= \log_{10}\left(100x^2\right)\\ &= \log_{10}(100) + \log_{10}(x^2)\\ &= 2 + 2\log_{10}(x)\end{aligned}$$

In terms of the redefined variables z and $H_1(z)$, we have

$$H_1(z) = 2 + 2z \tag{4}$$

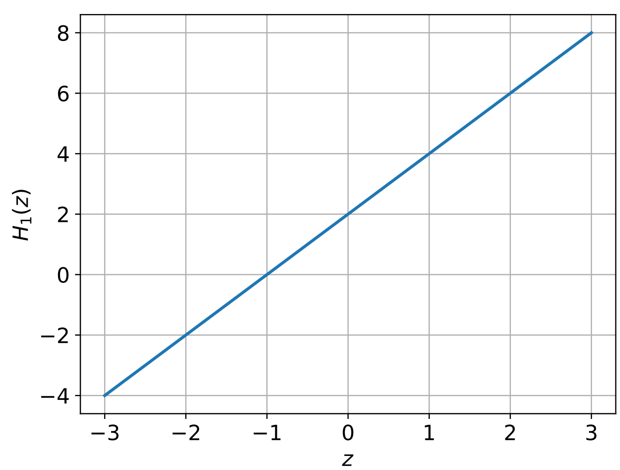


Figure 2: Plot of $H_1(z)$ against z.

Equation 4 represents a straight line with a slope of 2 and an intercept of 2. The resulting plot is shown in Figure 2.

b) $h(x) = g(x)$

Again, find a relationship between z and $H_2(z) \equiv \log_{10} h(x)$.

Hint: You might find that is not easy to directly evaluate $\log_{10} h(x)$. Try looking at different ranges of values of x. How can you approximate $h(x)$ when $x \ll 1$ (say around 10^{-5})? What will the approximation be when $x \gg 1$ (say around 10^5)?

Answer

We start with applying a logarithm to the expression provided to us

$$\begin{aligned}\log_{10} h(x) &= \log_{10}\left(10(1+x^{10})\right)\\ &= \log_{10}(10) + \log_{10}\left(1+x^{10}\right)\\ &= 1 + \log_{10}\left(1+x^{10}\right)\end{aligned}$$

From this point on, we don't seem to have an easy way to express $\log_{10}\left(1+x^{10}\right)$ in terms of $\log_{10}(x)$. In order to be able to do that, we first look at the case when $x \ll 1$. In this case, say when $x = 10^{-5}$,

$$\begin{aligned}1 + x^{10} &= 1 + (10^{-5})^{10}\\ &= 1 + 10^{-50}\end{aligned}$$

10^{-50} is a *very* small number when compared to 1. For all practical purposes, we can say that $1 + x^{10} \approx 1$ when $x \ll 1$. This give us

$$\log_{10} h(x) = 1 + \log_{10}\left(1+x^{10}\right) = 1 \tag{5}$$

Let us take the case when $x \gg 1$, say $x = 10^5$. In this case,

$$\begin{aligned}1 + x^{10} &= 1 + (10^{5})^{10}\\ &= 1 + 10^{50}\end{aligned}$$

10^{-50} is a *very* large number when compared to 1. For all practical purposes, we can say that $1 + x^{10} \approx x^{10}$ when $x \gg 1$. This give us

$$\log_{10} h(x) = 1 + \log_{10} x^{10} = 1 + 10\log_{10}(x) = 1 + 10z \tag{6}$$

Combining the results from Equations 5 and 6, we get

$$H_2(z) \approx \begin{cases} 1 & \text{if } z < 0 \\ 1 + 10z & \text{if } z > 0 \end{cases} \tag{7}$$

The domains change as $x < 1 \implies z < 0$ and similarly, $x > 1 \implies z > 0$. We also have an approximation because there is a grey area around $x = 1$ or $z = 0$ where our approximations do not hold very well.

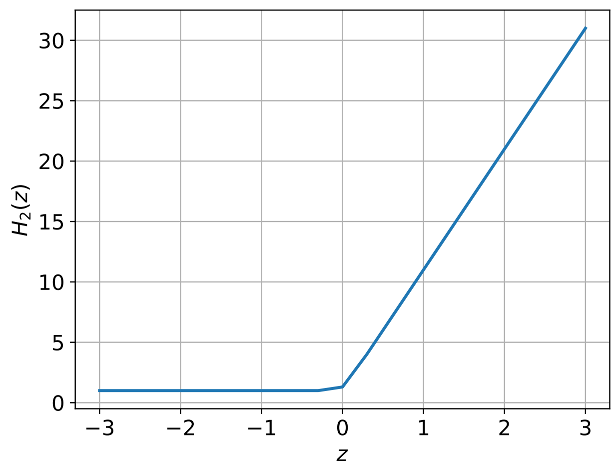


Figure 3: Plot of $H_2(z)$ against z.

Figure 3 show the plot of $H_2(z)$ against z without any approximations. Notice how close this is to the approximation we have derived in Equation 7.

c) $h(x) = f(x) \cdot g(x)$

First, write out the expression for $H_3(z) \equiv \log_{10} h(x)$ in terms of $H_1(z)$ and $H_2(z)$ that we have defined previously. Now use the different domains we had used in the previous problem to write out $H_3(z)$ for $x \ll 1$ and $x \gg 1$.

Answer

We start with the logarithm of $h(x)$. This gives us

$$\begin{aligned} H_3(z) &= \log_{10} h(x) \\ &= \log_{10} \left(f(x) \cdot g(x)\right) \\ &= \log_{10} f(x) + \log_{10} g(x) \\ &= H_1(z) + H_2(z) \end{aligned}$$

We can now plug in our expressions for $H_1(z)$ and $H_2(z)$.

$$H_3(z) \approx \begin{cases} 3 + 2z & \text{if } z < 0 \\ 3 + 12z & \text{if } z > 0 \end{cases} \tag{8}$$

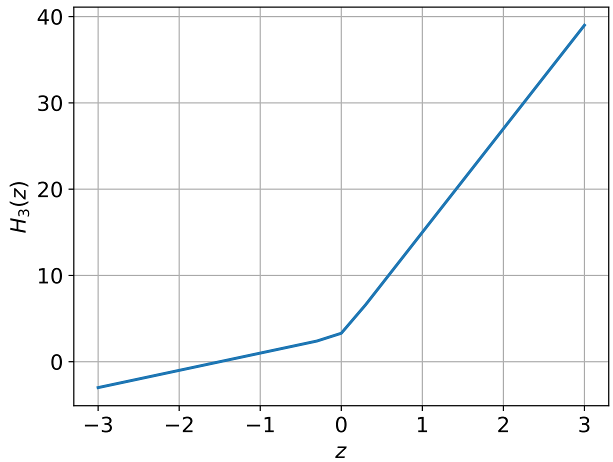


Figure 4: Plot of $H_3(z)$ against z.

Figure 4 shows the plot of $H_3(z)$ against z without making the approximations in Equation 9. Notice the change in slopes around $z = 0$. The intercepts from the previous functions $H_1(z)$ and $H_2(z)$ are also added together.

d) $h(x) = \frac{f(x)}{g(x)}$

Write out the expression for $H_4(x) \equiv \log_{10} h(x)$ in terms of $H_1(z)$ and $H_2(z)$ that we have defined previously. How can we simplify these for the two domains $x \ll 1$ and $x \gg 1$.

Answer

Like in the previous problem, we start with the logarithm of $h(x)$. This gives us

$$\begin{aligned} H_4(z) &= \log_{10} h(x) \\ &= \log_{10}\left(\frac{f(x)}{g(x)}\right) \\ &= \log_{10} f(x) - \log_{10} g(x) \\ &= H_1(z) - H_2(z) \end{aligned}$$

We can now plug in our expressions for $H_1(z)$ and $H_2(z)$.

$$H_3(z) \approx \begin{cases} 1 + 2z & \text{if } z < 0 \\ 1 - 8z & \text{if } z > 0 \end{cases} \tag{9}$$

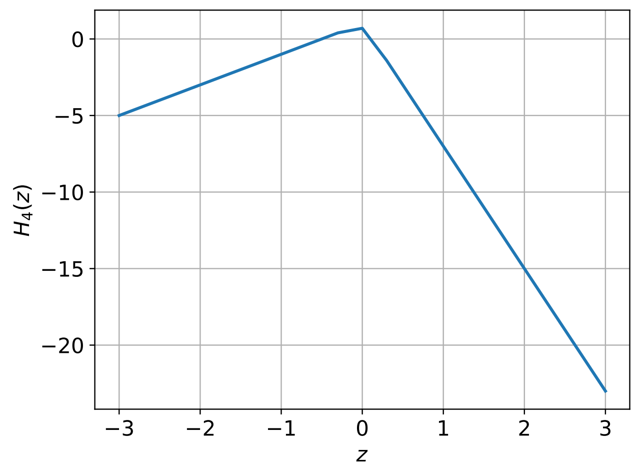
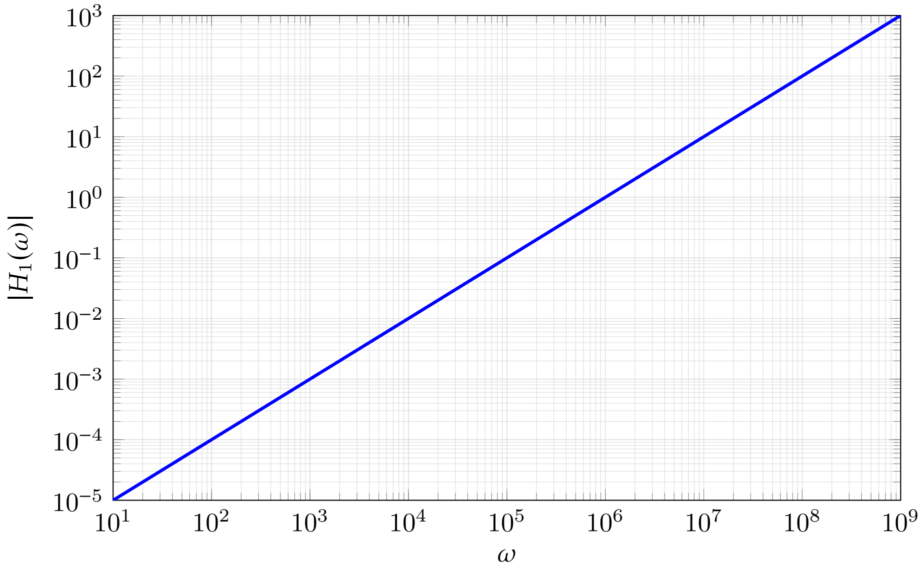


Figure 5: Plot of $H_4(z)$ against z.

Again here, we see that before $z = 0$, $H_4(z)$ increases with a slope of 2. After $z = 0$, the slope becomes negative. Notice how the slopes get subtracted after $z = 0$.

2 Bode Plot Practice

a) **Identify the locations of the poles and zeroes in the following magnitude Bode plot. What transfer function $H_1(\omega)$ would result in this plot?**

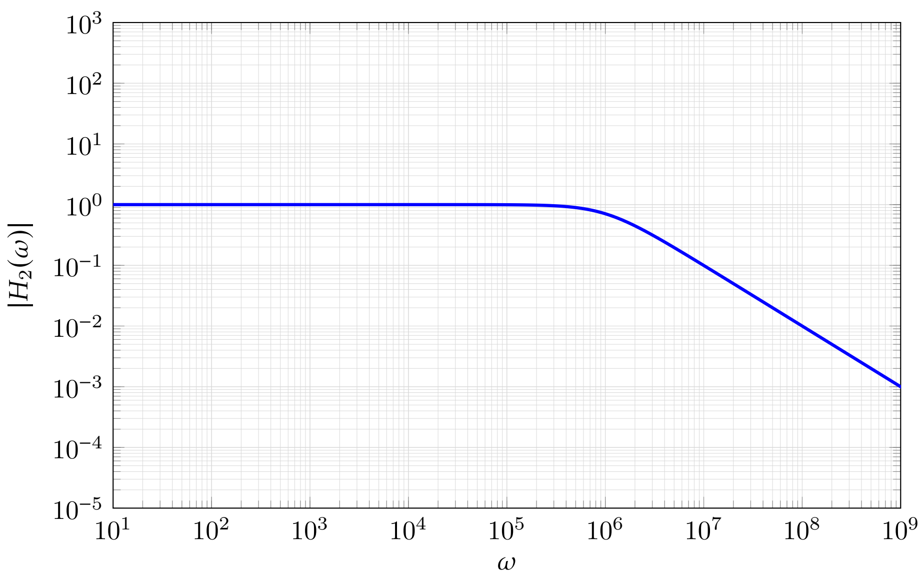


Answer

This is a zero at the origin, scaled by 10^{-6}. Notice how the scaling translated the original line instead of changing the slope, because of the logarithm property $\ln K\omega = \ln\omega + \ln K$.

$$H_1(\omega) = \frac{j\omega}{10^6}$$

b) **Identify the locations of the poles and zeroes in the following magnitude Bode plot. What transfer function $H_2(\omega)$ would result in this plot?**



Answer

There is a single pole at $\omega_p = 10^6 \frac{\text{rad}}{\text{s}}$. The transfer function is thus:

$$H_2(\omega) = \frac{1}{1 + \frac{j\omega}{10^6}}$$

This may also be recognized as the familiar form of a simple low pass filter.

c) **Identify the locations of the poles and zeroes in the following transfer function. Then sketch the magnitude Bode plot.**

$$H_3(\omega) = \frac{\frac{j\omega}{10^6}}{1 + \frac{j\omega}{10^6}}$$

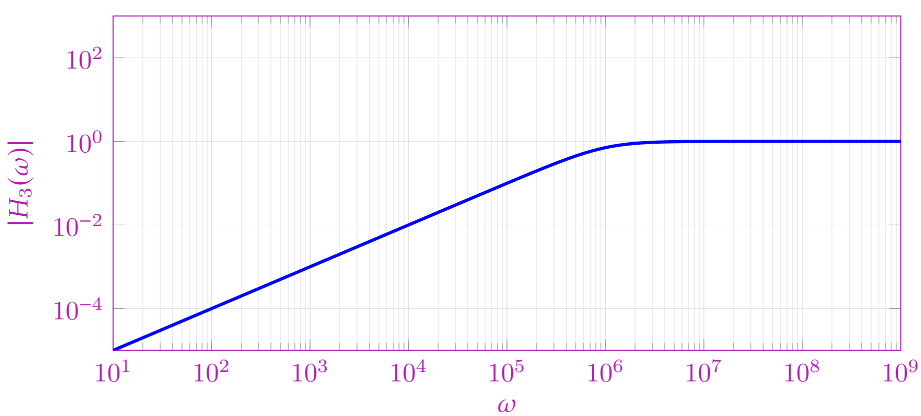
Answer

(1) Poles and Zeroes
There is a single zero at the origin ($\omega = 0$), and single pole at $\omega_{p1} = 10^6$.

(2) Constant K
Since $K = 10^{-6}$, and we have a zero at the origin, this means that our entire plot is shifted downward by 10^{-6}.

(3) Plotting
The plot should start with a slope of 1 since there is a zero at the origin. Since $K = 10^{-6}$, and there are no poles or zeros before 10^{-6}, we can say that $H(10^6) \approx 1$. Then we have a single pole at $\omega_{p1} = 10^6$ so this should cancel out the +1 slope from the zero and the magnitude plot will stay constant at 1.

There is a zero at the origin and a pole at $\omega_p = 10^6 \frac{\text{rad}}{\text{s}}$. The whole transfer function is scaled by 10^{-6} so that $|H_3(\omega >> 10^6)| \approx 1$ in the passband. This may also be recognized as the form of a simple high pass filter, created by cascading parts (a) and (b).



d) **Identify the locations of the poles and zeroes in the following magnitude Bode plot. Then sketch the magnitude Bode plot.**

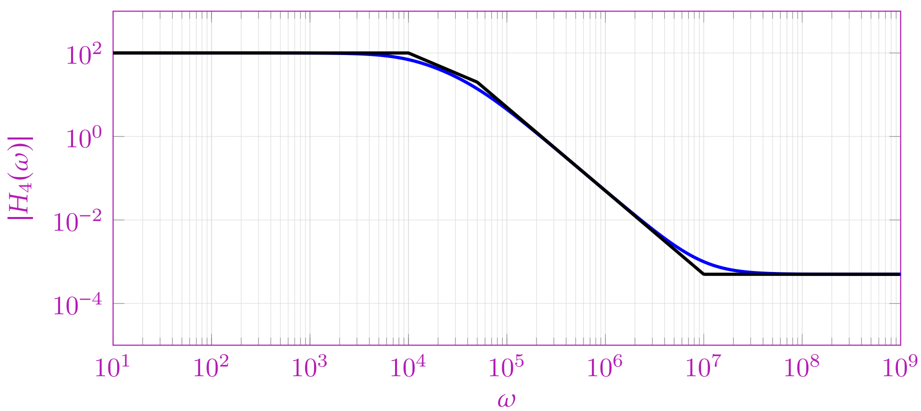
$$H_4(\omega) = 100 \frac{(1 + \frac{j\omega}{10^7})^2}{(1 + \frac{j\omega}{10^4})(1 + \frac{j\omega}{5 \times 10^4})}$$

Answer

(1) Poles and Zeroes
There is a double zero at $\omega_{z1} = 10^7$ and two poles at $\omega_{p1} = 10^4$ and $\omega_{p2} = 5 \cdot 10^4$.

(2) Constant K
Since $K = 100$, the entire plot should be shifted up by a factor of 100. However, since there are no zeros at the origin, this means that $H(0) = K = 100$ and our Bode plot will start at 100

(3) Plotting
The plot should start at 100. Then at $\omega_{p1} = 10^4$, it drops off with slope -1 and at the second pole ω_{p2} it drops off even quicker with slope -2. then at $\omega_{z1} = 10^7$, the double zero cancels out the -2 slope and the magnitude plot stays constant.



Since the two poles are spaced closely, it can be difficult to determine exactly where the second pole is located. As a result, Bode plots are often drawn using straight line approximations (depicted in the above figure). For many practical applications, this is sufficient and has the added benefit that changes in slope are sharply delineated, making it easier to identify frequencies of interest. However, it is important to note that ultimately straight lines are an approximation and in certain cases they can fail to convey crucial information, with one particularly relevant example being resonant RLC circuits.