1 Transfer Functions

When analyzing circuits in the phasor domain, we have always told you what the input voltage to the circuit was. However, sometimes we have many input sinusoids and we would like to see how the circuit generically responds to a sinusoid input of frequency ω. We want to see how an input sinusoid "transfers" into an output sinusoid. How do we do this?

Let's start with a simple RC circuit.

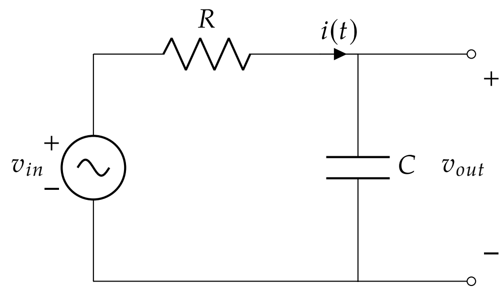


Figure 1: First Order RC Low Pass Filter

In the phasor domain, the impedance of the capacitor is $Z_C = \frac{1}{j\omega C}$ and the impedance of the resistor is $Z_R = R$. Because we treat impedances the same as resistances, this circuit looks like a voltage divider in the phasor domain. Remember we must also represent v_{in} as a phasor V_{in}; transfer functions are in the phasor domain only, not the time domain.

$$V_{out} = \frac{Z_C}{Z_R + Z_C} V_{in} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} V_{in} = \frac{1}{j\omega RC + 1} V_{in}$$

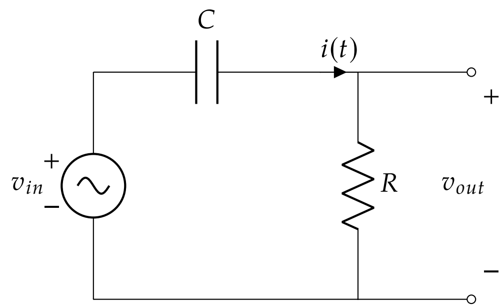
We define the frequency response as

$$H(\omega) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + j\omega RC}$$

Now, given an arbitrary input sinusoid, if we multiply it by the frequency response, we can get the output sinusoid. What this allows us to do is to model any arbitrary circuit as a single-input-single-output black box. The transfer function completely defines how our circuit works. We'll take a look at a couple of examples in the discussion to understand what this means.

2 Circuit Filters

Let's start with a simple RC circuit with a sinusoidal input $v_{in}(t)$.



For this question, we will define the transfer function $H(\omega)$ as the ratio of the input and output phasors

$$H(\omega) = \frac{V_{out}}{V_{in}} \tag{1}$$

The resistor $R = 1\,\text{k}\Omega$ and the capacitor $C = 1\,\mu\text{F}$.

a) Compute the transfer function $H(\omega)$.

Answer

$$H(\omega) = \frac{V_{out}}{V_{in}} = \frac{Z_R}{Z_R + Z_C} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{1 + j\omega RC}$$

b) What are the magnitude and phase of the transfer function $H(\omega)$?

Answer

Remember that the transfer function like phasors, is also a complex number.

$$H(\omega) = \frac{j\omega RC}{1 + j\omega RC} \tag{2}$$

To find the magnitude of the quotient of two complex numbers, we can divide the magnitudes of the numerator and denominator

$$\left|H(\omega)\right| = \frac{\left|j\omega RC\right|}{\left|1 + j\omega RC\right|} = \frac{\omega RC}{\sqrt{1^2 + (\omega RC)^2}} \tag{3}$$

Similarly to find the phase of $H(\omega)$, we can subtract the phases of the numerator and denominator

$$\angle H(\omega) = \angle j\omega RC - \angle(1 + j\omega RC) = \frac{\pi}{2} - \tan^{-1}(\omega RC) \tag{4}$$

c) What happens to the magnitude and phase of $\omega \to 0$? How about when $\omega \to \infty$? Lastly, explain why this circuit is often called a *high-pass filter*.

Answer

For small values of ω close to zero we see that $\left|H(\omega)\right| \approx 0$ and $\angle H(\omega) \approx \frac{\pi}{2}$.

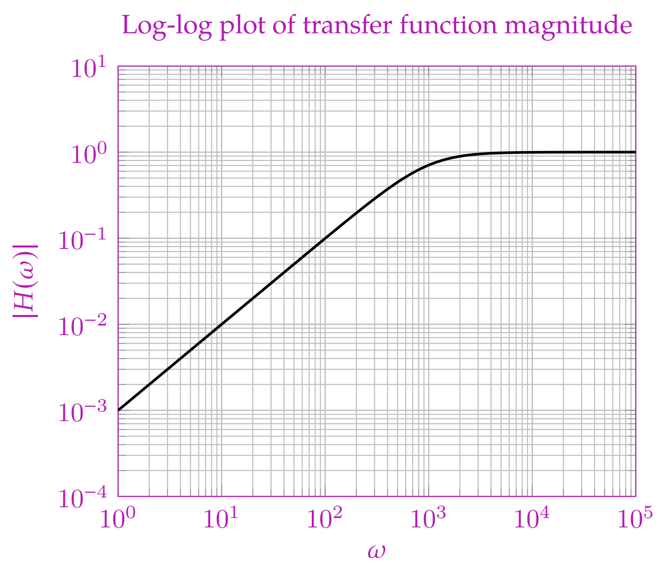
On the other hand, for $\omega \to \infty$, we see that $\left|H(\omega)\right| \approx 1$ and $\angle H(\omega) \approx 0$.

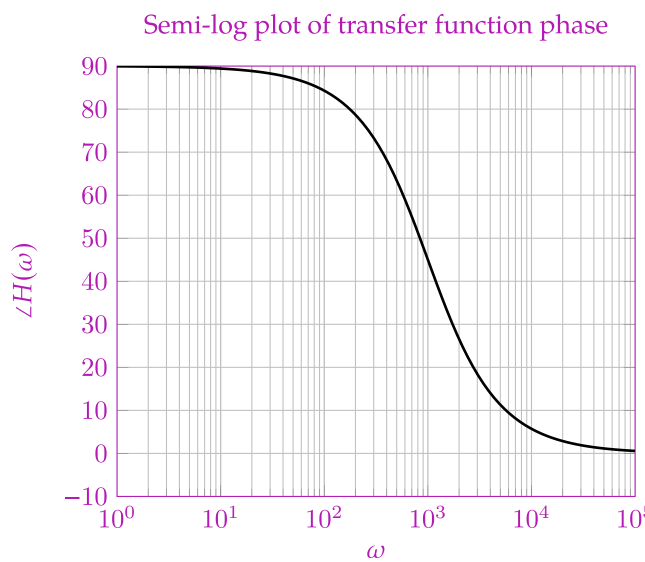
This circuit is called a high-pass filter since it lets higher frequency sinusoids through while attenuating lower frequency sinusoids.

d) Plot the magnitude and phase of $H(\omega)$ using a computer.

Answer

Plotting the expressions for $\left|H(\omega)\right|$ and $\angle H(\omega)$, on a log-log scale, we see that





e) Now let's take a look at a couple of outputs given some input sinusoids. Let $R = 1\,\text{k}\Omega$ and $C = 1\,\mu\text{F}$. Compute the steady state output voltage for the following inputs

i) $v_{in}(t) = 4\cos\left(1000t + \frac{\pi}{3}\right)$ [V]

ii) $v_{in}(t) = 2\sin\left(10^6 t - \frac{\pi}{4}\right)$ [V]

iii) $v_{in}(t) = 5\,\text{V}$

Answer

The key-idea is to identify the frequency of the signal v_{in} and use the transfer function $H(\omega)$

i) We first note that $\omega = 1000$ and $V_{in} = 4e^{j\frac{\pi}{3}}$. Evaluating $H(\omega)$ at $\omega = 1000$, we see that

$$\left|H(1000)\right| = \frac{1}{\sqrt{2}} \qquad \angle H(1000) = \frac{\pi}{4}$$

Therefore, since $V_{out} = H(1000) \cdot V_{in}$, it follows that $V_{out} = 2\sqrt{2}e^{j\frac{7\pi}{12}}$. Lastly, taking the inverse phasor transform, we $v_{out}(t) = 2\sqrt{2}\cos\left(1000t + \frac{7\pi}{12}\right)$.

ii) Note that $\omega = 10^6$. To write v_{in} as a phasor, we must first change it to a cosine reference

$$\begin{aligned} v_{in}(t) &= 2\sin\left(10^6 t - \frac{\pi}{4}\right) \\ &= 2\cos\left(10^6 t - \frac{\pi}{4} - \frac{\pi}{2}\right) \\ &= 2\cos\left(10^6 t - \frac{3\pi}{4}\right) \end{aligned}$$

This tells us that $V_{in} = 2e^{-j\frac{3\pi}{4}}$. Evaluating $H(10^6)$, we see that

$$\left|H(10^6)\right| = 0.9999995 \approx 1 \quad \angle H(10^6) = 0.001 \approx 0$$

This tells us that V_{out} is approximately equal to V_{in}. Therefore, $v_{out}(t) = 2\cos\left(10^6 t - \frac{3\pi}{4}\right)$.

iii) A constant voltage has zero frequency meaning $\omega = 0$. The phasor will be $V_{in} = 5e^{j\cdot 0}$. Evaluating $H(\omega)$ at $\omega = 0$, we see that $\left|H(0)\right| = 0$ and $\angle H(0) = \frac{\pi}{2}$. Therefore, $V_{out} = 0$ which implies that $v_{out}(t) = 0\,\text{V}$.

3 Rational Transfer Function Form

When we write the transfer function of an arbitrary circuit, it always takes the following form. This is called a "rational transfer function." We also like to factor the numerator and denominator, so that they become easier to work with and plot:

$$H(\omega) = \frac{z(\omega)}{p(\omega)} = \frac{(j\omega)^{N_{z0}}}{(j\omega)^{N_{p0}}}\left(\frac{(j\omega)^n\alpha_n + (j\omega)^{n-1}\alpha_{n-1} + \cdots + j\omega\alpha_1 + \alpha_0}{(j\omega)^m\beta_m + (j\omega)^{m-1}\beta_{m-1} + \cdots + j\omega\beta_1 + \beta_0}\right)$$

$$= K\frac{(j\omega)^{N_{z0}}\left(1 + j\frac{\omega}{\omega_{z1}}\right)\left(1 + j\frac{\omega}{\omega_{z2}}\right)\cdots\left(1 + j\frac{\omega}{\omega_{zn}}\right)}{(j\omega)^{N_{p0}}\left(1 + j\frac{\omega}{\omega_{p1}}\right)\left(1 + j\frac{\omega}{\omega_{p2}}\right)\cdots\left(1 + j\frac{\omega}{\omega_{pm}}\right)}$$

Here, we define the constants ω_z as "zeros" and ω_p as "poles", and N_{z0}, N_{p0} are the number of zeros and poles at $\omega = 0$

4 Poles and Zeros

Identify all of the zeros and poles for the following transfer functions

a) $H_1(\omega) = \dfrac{1}{1 + j\omega/10}$

Answer

No zeros, single pole at $\omega_p = 10$.

b) $H_2(\omega) = \dfrac{1 + j\omega/50}{(1 + j\omega/100)(1 + j\omega/5)}$

Answer

This transfer function is already in rational form. We have a zero at $\omega_z = 50$ and two poles. One at $\omega_{p1} = 100$ and another at $\omega_{p2} = 5$.

c) $H_3(\omega) = \dfrac{j\omega \cdot 100}{1 + j\omega \cdot 1000}$

Answer

Notice how the $j\omega$ terms are **multiplied** by constants instead of dividing. Putting this transfer function into rational form, it should be

$$H_2(\omega) = 100\frac{j\omega}{1 + j\omega/10^{-3}}$$

Therefore, we have a single zero at the origin, and a single pole at $\omega_p = 10^{-3}$.

d) $H_4(\omega) = \dfrac{20 + j\omega \cdot 40}{(1 + j\omega \cdot 10)^2}$

Answer

We first put this transfer function in rational form:

$$H_4(\omega) = 20\frac{1 + j\omega/(0.5)}{(1 + j\omega/(0.1))^2}$$

Therefore, we have a zero at $\omega_z = 0.5$, and two poles, commonly referred to as a "double pole" at $\omega_p = 0.1$.

e) $H_5(\omega) = \dfrac{10 + j\omega}{1000 + 110j\omega + (j\omega)^2}$

Answer

We first put this transfer function in rational form:

$$\begin{aligned} H_5(\omega) &= 10\frac{1 + j\omega/10}{(100 + j\omega)(10 + j\omega)} \\ &= \frac{10}{1000}\frac{1 + j\omega/10}{(1 + j\omega/100)(1 + j\omega/10)} \end{aligned}$$

Therefore, we have a zero at $\omega_z = 10$, a pole at $\omega_{p1} = 10$ and another pole at $\omega_{p2} = 100$.