1 Complex Numbers

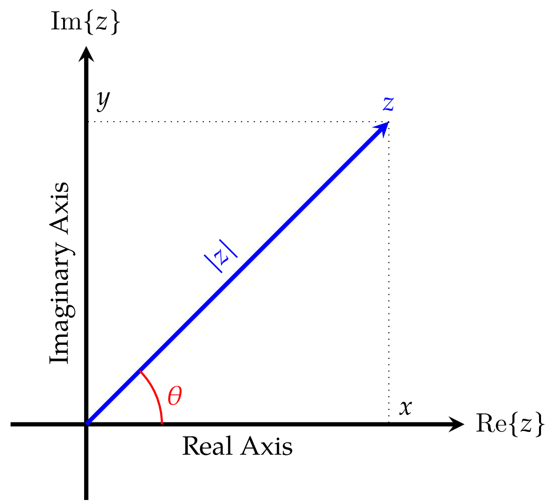

Figure 1: Complex number z represented as a vector in the complex plane.

A complex number z is an ordered pair (x, y), where x and y are real numbers, written as $z = x + jy$ where $j = \sqrt{-1}$. A complex number can also be written in polar form as follows:

$$z = |z|e^{j\theta}$$

where $|z|$ is the maginutude of z, given by

$$|z| = \sqrt{x^2 + y^2}.$$

The phase or argument of a complex number is denoted as θ and is given by

$$\theta = \text{atan2}(y, x).$$

Here, $\text{atan2}(y, x)$ is a function that returns the angle from the positive x-axis to the vector from the origin to the point (x, y) [1].

The complex conjugate of a complex number z is denoted by $\bar{z}$ (or might also be written z^*) and is given by

$$\bar{z} = x - jy.$$

From this we see that $|z|^2 = x^2 + y^2 = z \cdot \bar{z}$.

Euler's Identity is

$$e^{j\theta} = \cos(\theta) + j\sin(\theta).$$

With this definition, the polar representation of a complex number will make more sense. Note that

$$|z|e^{j\theta} = |z|\cos(\theta) + j\,|z|\sin(\theta).$$

The reason for these definitions is to exploit the geometric interpretation of complex numbers, as illustrated in Figure 1, in which case $|z|$ is the magnitude and $e^{j\theta}$ is the unit vector that defines the direction.

2 Useful Identities

[1]See its relation to $\tan^{-1}\left(\frac{y}{x}\right)$ at `https://en.wikipedia.org/wiki/Atan2`.

Complex Number Properties

Rectangular vs. polar forms: $z = x + jy = |z|e^{j\theta}$

where $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$, $\theta = \text{atan2}(y, x)$. We can also write $x = |z|\cos\theta$, $y = |z|\sin\theta$.

Euler's identity: $e^{j\theta} = \cos\theta + j\sin\theta$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}, \ \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Complex conjugate: $\bar{z} = x - jy = |z|e^{-j\theta}$

$$\overline{(z + w)} = \bar{z} + \bar{w}, \ \overline{(z - w)} = \bar{z} - \bar{w}$$

$$\overline{(zw)} = \bar{z}\bar{w}, \ \overline{(z/w)} = \bar{z}/\bar{w}$$

$$\bar{z} = z \Leftrightarrow z \text{ is real}$$

$$\overline{(z^n)} = (\bar{z})^n$$

Complex Algebra

Let $z_1 = x_1 + jy_1 = |z_1|e^{j\theta_1}$, $z_2 = x_2 + jy_2 = |z_2|e^{j\theta_2}$. (Note that we adopt the easier representation between rectangular form and polar form.)

Addition: $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

Multiplication: $z_1 z_2 = |z_1||z_2|e^{j(\theta_1+\theta_2)}$

Division: $\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}e^{j(\theta_1-\theta_2)}$

Power: $z_1^n = |z_1|^n e^{jn\theta_1}$

$$z_1^{\frac{1}{2}} = \pm|z_1|^{\frac{1}{2}} e^{j\frac{\theta_1}{2}}$$

Useful Relations

$$\begin{aligned}
-1 &= j^2 = e^{j\pi} = e^{-j\pi} \\
j &= e^{j\frac{\pi}{2}} = \sqrt{-1} \\
-j &= -e^{j\frac{\pi}{2}} = e^{-j\frac{\pi}{2}} \\
\sqrt{j} &= (e^{j\frac{\pi}{2}})^{\frac{1}{2}} = \pm e^{j\frac{\pi}{4}} = \frac{\pm(1+j)}{\sqrt{2}} \\
\sqrt{-j} &= (e^{-j\frac{\pi}{2}})^{\frac{1}{2}} = \pm e^{-j\frac{\pi}{4}} = \frac{\pm(1-j)}{\sqrt{2}}
\end{aligned}$$

3 Complex Algebra

a) Express the following values in polar forms: -1, j, $-j$, $\sqrt{j}$, and $\sqrt{-j}$. Recall $j^2 = -1$.

Answer

A complex number can be represented in the following forms:

$$z = a + jb = rcos(\theta) + jrsin(\theta) = re^{j\theta}, \tag{1}$$

where, $r = \sqrt{a^2 + b^2}$, $\theta = tan^{-1}\left(\frac{b}{a}\right)$ and a, b are real numbers.

$$\begin{aligned}
-1 &= j^2 = e^{j\pi} = e^{-j\pi} \\
j &= e^{j\frac{\pi}{2}} = \sqrt{-1} \\
-j &= -e^{j\frac{\pi}{2}} = e^{-j\frac{\pi}{2}} \\
\sqrt{j} &= (e^{j\frac{\pi}{2}})^{\frac{1}{2}} = e^{j\frac{\pi}{4}} = \frac{1+j}{\sqrt{2}} \\
\sqrt{-j} &= (e^{-j\frac{\pi}{2}})^{\frac{1}{2}} = e^{-j\frac{\pi}{4}} = \frac{1-j}{\sqrt{2}}
\end{aligned}$$

b) Represent $\sin\theta$ and $\cos\theta$ using complex exponentials. (*Hint:* Use Euler's identity $e^{j\theta} = \cos\theta + j\sin\theta$.)

Answer

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}, \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}.$$

c) For complex number $z = x + jy$ show that $|z| = \sqrt{z\bar{z}}$, where $\bar{z}$ is the complex conjugate of z.

Answer

We can follow the definition of complex conjugate and magnitude:

$$\sqrt{z\bar{z}} = \sqrt{(x + jy)(x - jy)} = \sqrt{x^2 + y^2} = |z|$$

For the next two parts, let $a = 1 - j\sqrt{3}$ and $b = \sqrt{3} + j$.

d) Express a and b in polar form.

Answer

Following the definitions in part a:

$$|a| = 2, \ \ |b| = 2, \ \ \theta_a = -\frac{\pi}{3}, \ \ \theta_b = \frac{\pi}{6}.$$

Hence,

$$a = 2e^{-j\frac{\pi}{3}} \quad b = 2e^{j\frac{\pi}{6}}.$$

e) Find ab, $a\bar{b}$, $\frac{a}{b}$, $a + \bar{a}$, $a - \bar{a}$, $\overline{ab}$, $\bar{a}\bar{b}$, and $\sqrt{b}$.

Answer

$$ab = 4 \cdot e^{-j\frac{\pi}{6}} = 2\sqrt{3} - 2j$$
$$a\bar{b} = 4 \cdot e^{-j\frac{\pi}{2}} = -4j$$
$$\frac{a}{b} = e^{-j\frac{\pi}{2}} = -j$$
$$a + \bar{a} = 2$$

Note, $a + \bar{a}$ is a purely real number.

$$a - \bar{a} = -2j\sqrt{3}$$

Note, $a - \bar{a}$ is a purely imaginary number.

$$\overline{ab} = 2\sqrt{3} + 2j$$
$$\bar{a}\bar{b} = (1 + j\sqrt{3})(\sqrt{3} - j) = \sqrt{3} + \sqrt{3} + j(3 - 1) = 2\sqrt{3} + 2j$$

Note, $\overline{ab} = \bar{a}\bar{b}$.

$$\sqrt{b} = \sqrt{2} \cdot e^{j\frac{\pi}{12}}$$

f) Show the number a in complex plane, marking the distance from origin and angle with real axis.

Answer

Location of a in the complex plane is shown in the following figure.

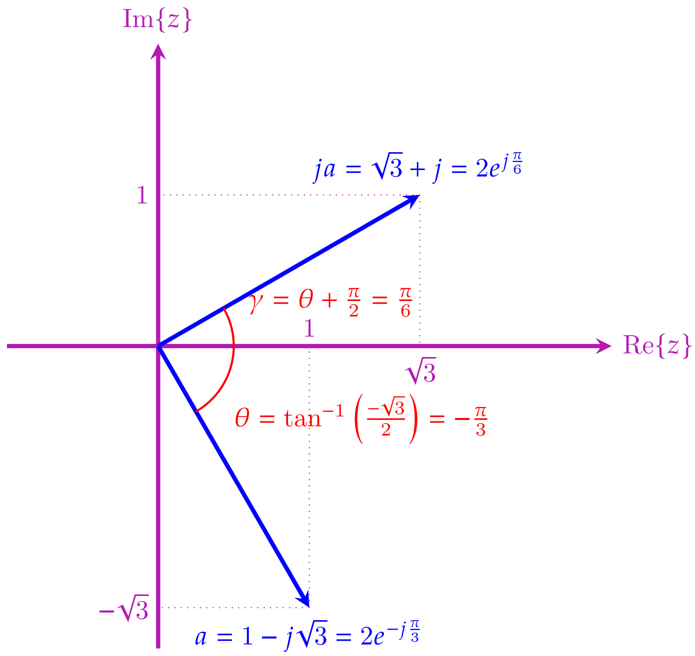

Figure 2: Complex number a represented as a vector in the complex plane.

g) Show that multiplying a with j is equivalent to rotating the magnitude of the complex number by $\pi/2$ or 90 degrees in the complex plane.

Answer

Multiplying a by j:

$$ja = e^{j\pi/2} \times 2e^{-j\pi/3} = 2e^{j\pi/6} = \sqrt{3} + j \tag{2}$$

The rotation is demonstrated in the complex plane plot.

4 Change of basis

Recall from Discussion 3A that we ended up with the following differential equations. We can represent the differential equation as follows with $x_1 = V_{C1}$, $x_2 = V_{C2}$, and $V_{in} = 0$. The initial condition is $x_1(0) = 7, x_2(0) = 7$.

$$\frac{\mathrm{d}}{\mathrm{d}t} x_1(t) = -5x_1(t) + 2x_2(t) \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} x_2(t) = 6x_1(t) - 6x_2(t) \tag{4}$$

We can rewrite the above differential equations as a vector differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}t} \vec{x}(t) = A\vec{x}(t), \tag{5}$$

where $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $A = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}$. And the diagonalization of A writes

$$A = V\Lambda V^{-1} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}. \tag{6}$$

Below we would like to solve the above differential equations using change of basis to the eigenbasis. In fact this is what we have been doing the whole time to solve vector differential equations using diagonalization. The following questions will make this clear for you.

To review the concept of a basis, any vector $\vec{u}$ can be written as a sum of linear combination of the basis. In the following figure, the standard basis (e_1, e_2) and the eigenbasis (v_1, v_2) are shown.

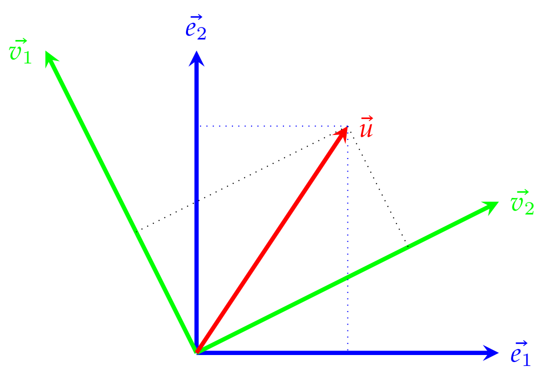

Figure 3: Any point in the 2-d plane can be spanned by both bases, $(\vec{e}_1, \vec{e}_2)$ and $(\vec{v}_1, \vec{v}_2)$. Figure not to scale.

a) Consider a general vector $\begin{bmatrix} a \\ b \end{bmatrix}$, can you find the coordinate of the vector relative to the standard basis $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$? That is, write the vector in the following form with x_1 and x_2 being its coordinates.

$$\begin{bmatrix} a \\ b \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Answer

The answer is quite straight forward, we have $x_1 = a$ and $x_2 = b$ as the coordinate of the vector relative to the standard basis.

b) This time we consider another basis $\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. (this is in fact the eigenbasis of A as was given in the diagonalization) Can you find the coordinate of the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ relative to the eigenbasis? That is, write the vector in the following form with z_1 and z_2 being the coordinates. What can you find based on the answer to previous part?

$$\begin{bmatrix} a \\ b \end{bmatrix} = z_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + z_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Answer

The expression is equivalent to

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = V \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Thus the solution is given by left multiplying both sides with V^{-1} which is given in the diagonalization.

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = V^{-1} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\frac{3}{7}a + \frac{2}{7}b \\ \frac{2}{7}a + \frac{1}{7}b \end{bmatrix}$$

Plugging in $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we have the following change of basis from the coordinates relative to standard basis to the coordinates relative to eigenbasis.

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = V^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = V \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

c) Now express the differential equations in terms of the coordinates relative to the eigenbasis v_1 and v_2 by plugging in your answer to the previous part to the differential equation (5). Use $\vec{z}(t) = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ to represent the new variable (coordinate) you arrive at.

Answer

The differential equation writes

$$\frac{\mathrm{d}}{\mathrm{d}t} \vec{x}(t) = A\vec{x}(t).$$

Substituting in the diagonalization of A, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \vec{x}(t) = V \Lambda V^{-1} \vec{x}(t).$$

Left multiply both sides with V^{-1},

$$\frac{\mathrm{d}}{\mathrm{d}t} V^{-1} \vec{x}(t) = \Lambda V^{-1} \vec{x}(t).$$

Plugging in the answer to previous part and change the basis, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}(t) = \Lambda\vec{z}(t), \tag{7}$$

where $\vec{z}(t) = V^{-1}\vec{x}(t)$ is the coordinate relative to the eigenbasis.

d) Solve the differential equation with $\vec{z}(t)$ and find the solution for $\vec{x}(t)$.

Answer

The initial condition is given by

$$\vec{z}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Solving differential equation (7), e based on the form of the problem and our previous differential equation experience:

$$\vec{z}(t) = \begin{bmatrix} K_1 e^{-9t} \\ K_2 e^{-2t} \end{bmatrix}$$

Plugging in for the initial condition gives:

$$\vec{z}(t) = \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix}$$

Now convert the solution back to the original coordinate to find $\vec{x}(t)$:

$$\vec{x}(t) = V\vec{z}(t) = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-9t} + 6e^{-2t} \\ -2e^{-9t} + 9e^{-2t} \end{bmatrix}$$