1 Discrete Time Systems

Consider a discrete-time system with $x[n]$ as input and $y[n]$ as output.

$$
\begin{align*}
  x[n] & \rightarrow \boxed{} \rightarrow y[n]
\end{align*}
$$

The following are some of the possible properties that a system can have:

**Linearity**

A linear system has the properties below:

1. **additivity**
   $$
   x_1[n] + x_2[n] \rightarrow \boxed{} \rightarrow y_1[n] + y_2[n] \tag{1}
   $$

2. **scaling** (or homogeneity)
   $$
   ax[n] \rightarrow \boxed{} \rightarrow ay[n] \tag{2}
   $$

   Here, $\alpha$ is some constant.

   Together, these two properties are known as **superposition**:

   $$
   a_1 x_1[n] + a_2 x_2[n] \rightarrow \boxed{} \rightarrow a_1 y_1[n] + a_2 y_2[n]
   $$

**Time Invariance**

A system is time-invariant if its behavior is fixed over time:

$$
\begin{align*}
  x[n - n_0] & \rightarrow \boxed{} \rightarrow y[n - n_0] \tag{3}
\end{align*}
$$

**Causality**

A causal system has the property that $y[n_0]$ only depends on $x[n]$ for $n \in (-\infty, n_0]$. An intuitive way of interpreting this condition is that the system does not “look ahead.”

**Bounded-Input, Bounded-Output (BIBO) Stability**

In a BIBO stable system, if $x[n]$ is bounded, then $y[n]$ is also bounded. A signal $x[n]$ is bounded if there exists an $M$ such that $|x[n]| \leq M < \infty \ \forall n.$
2  Linear Time-Invariant (LTI) Systems

A system is LTI if it is both linear and time-invariant. We define the impulse response of an LTI system as the output \( h[n] \) when the input \( x[n] = \delta[n] \) where \( \delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \).

An LTI system can be uniquely characterized by its impulse response \( h[n] \). In addition, the following properties hold:

- An LTI system is causal iff \( h[n] = 0 \quad \forall n < 0 \).
- An LTI system is BIBO stable iff its impulse response is absolutely summable:

\[
\sum_{n=-\infty}^{\infty} |h[n]| < \infty
\]

Convolution Sum

Consider the following LTI system with impulse response \( h[n] \):

\[
\begin{array}{ccc}
x[n] & \rightarrow & \square \\
& \rightarrow & y[n]
\end{array}
\]

Notice that we can write \( x[n] \) as a sum of impulses:

\[
x[n] = \ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \ldots = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]
\]

In addition, we know that:

\[
\begin{array}{ccc}
\delta[n] & \rightarrow & \square \\
& \rightarrow & h[n]
\end{array}
\]

By applying the LTI property of our system, we get that

\[
x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow \square \rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]
\]

The expression \( \sum_{k=-\infty}^{\infty} x[k]h[n-k] \) is referred to as the convolution sum and can be written as \( x[n] * h[n] \) or \( (x * h)[n] \).
Determine if the following systems are linear, time-invariant, and/or causal.

(a) $y[n] = 2x[-2 + 3n] + 2x[2 + 3n]$

**Answer**

linear, not time-invariant, not causal

- **Linearity:** Set the input to $\hat{x}[n] = a_1 x_1[n] + a_2 x_2[n]$. Then

  $$
  \hat{y}[n] = 2\hat{x}[-2 + 3n] + 2\hat{x}[2 + 3n] \\
  = 2(a_1 x_1[-2 + 3n] + a_2 x_2[-2 + 3n]) + 2(a_1 x_1[2 + 3n] + a_2 x_2[2 + 3n]) \\
  = 2a_1 x_1[-2 + 3n] + 2a_1 x_1[2 + 3n] + 2a_2 x_2[-2 + 3n] + 2a_2 x_2[2 + 3n] \\
  = a_1 \left(2x_1[-2 + 3n] + 2x_1[2 + 3n]\right) + a_2 \left(2x_2[-2 + 3n] + 2x_2[2 + 3n]\right)
  $$

- **Time invariance:**

  Let $\hat{x}[n] = x[n - n_0]$ be a delayed input signal. Then, the corresponding output $\hat{y}[n]$ is equal to $2x[-2 + 3n - n_0] + 2x[2 + 3n - n_0]$. However, we can see that $\hat{y}[n] \neq y[n - n_0] = 2x[-2 + 3(n - n_0)] + 2x[2 + 3(n - n_0)]$

- **Causality:** Note that $y[0] = 2x[-2] + 2x[2]$ depends on $x[2]$, so the system is not causal.

(b) $y[n] = 4^{x[n]}$

**Answer**

non-linear, time-invariant, causal

- **Linearity:** Let $\hat{x}[n] = 2x[n]$. Then $\hat{y}[n] = 16^{\hat{x}[n]} \neq 2y[n]$. Thus, the system is not linear.

- **Time invariance:** Let $\hat{x}[n] = x[n - n_0]$. Then $\hat{y}[n] = 4^{\hat{x}[n]} = 4^{x[n-n_0]} = y[n - n_0]$, so the system is time-invariant.

- **Causality:** Note that $y[n_0]$ depends on $x[n_0]$ only, and not on any $x[n]$ with $t > n_0$. The system is therefore causal.

**Additional practice:**

(c) $y[n] - y[n - 1] = x[n] - x[n - 1] - x[n - 2]$
Answer

linear, time-invariant, causal

- Linearity: Let $x_1[n]$ and $x_2[n]$ be inputs with corresponding outputs $y_1[n]$ and $y_2[n]$ Set the input to $\hat{x}[n] = a_1x_1[n] + a_2x_2[n]$. Then

\[
\begin{align*}
\hat{y}[n] - \hat{y}[n-1] &= \hat{x}[n] - \hat{x}[n-1] - \hat{x}[n-2] \\
&= (a_1x_1[n] + a_2x_2[n]) - (a_1x_1[n-1] + a_2x_2[n-1]) - (a_1x_1[n-2] + a_2x_2[n-2]) \\
&= a_1(x_1[n] - x_1[n-1] - x_1[n-2]) + a_2(x_2[n] - x_2[n-1] - x_2[n-2]) \\
&= a_1(y_1[n] - y_1[n-1]) + a_2(y_2[n] - y_2[n-1])
\end{align*}
\]

Note that this is true for all $n$.

- Time invariance: Consider input $x[n]$ and corresponding output $y[n]$ Let $\hat{x}[n] = x[n - n_0]$. The corresponding output $\hat{y}[n]$ follows

\[
\begin{align*}
\hat{y}[n] - \hat{y}[n-1] &= \hat{x}[n] - \hat{x}[n-1] - \hat{x}[n-2] \\
&= x[n - n_0] - x[n - n_0 - 1] - x[n - n_0 - 2] \\
&= y[n - n_0] - y[n - n_0 - 1].
\end{align*}
\]

Therefore, the system is time-invariant.

- Causality: Note that $y[n_0] - y[n_0 - 1]$ depends only on $x[n_0], x[n_0 - 1], x[n_0 - 2]$, and $n_0, n_0 - 1, n_0 - 2 < n_0$ so the system is causal (no output depends on a future input).

d) $y[n] = x[n] + nx[n - 1]$

Answer

linear, not time-invariant, causal

- Linearity: Let $x_1[n]$ and $x_2[n]$ be inputs with corresponding outputs $y_1[n]$ and $y_2[n]$ Set the input to $\hat{x}[n] = a_1x_1[n] + a_2x_2[n]$. We check the system is linear:

\[
\begin{align*}
\hat{y}[n] &= \hat{x}[n] + nx[n-1] \\
&= a_1x_1[n] + a_2x_2[n] + n(a_1x_1[n-1] + a_2x_2[n-1]) \\
&= a_1(x_1[n] + nx_1[n-1]) + a_2(x_2[n] + nx_2[n-1]) \\
&= a_1y_1[n] + a_2y_2[n].
\end{align*}
\]

- Time invariance: Let $x[n]$ be an input with output $y[n]$. Set $\hat{x}[n] = x[n - n_0]$ for $n_0 \neq 0$ and note that $y[n - n - 0] = x[n - n_0] + (n - n_0)x[n - n_0 - 1]$ but $\hat{y}[n] = \hat{x}[n] + nx[n-1] = x[n - n_0] + nx[n - n_0 - 1]$. Therefore $\hat{y}[n] \neq y[n-n_0]$, and the system is not time invariant.

- Causality: Observe that $y[n_0]$ only depends on $x$ at $n_0$ and $n_0 - 1$, so it does not depend on any future $x$. The system is causal.

e) $y[n] = 2^n \cos(x[n])$
Answer

not linear, not time-invariant, causal

• Linearity: Suppose $x[n]$ is an input with corresponding output $y[n]$, and let $\hat{x}[n] = 2x[n]$. Then

\[
\hat{y}[n] = 2^n \cos(\hat{x}[n]) \\
= 2^n \cos(2x[n]) \\
\neq 2(2^n) \cos x[n].
\]

Therefore, the system is not linear.

• Time invariance: Consider the sequence $\hat{x}[n] = -x[n - n_0]$ for some $n_0 \neq 0$. Note that

\[
\hat{y}[n] = 2^n \cos(\hat{x}[n]) \\
= 2^n \cos(x[n - n_0]) \\
\neq 2^{n-n_0} \cos(x[n - n_0]) = y[n - n_0].
\]

This shows the system is not time invariant.

• Causality: Observe that $y[n_0]$ does not depend on any inputs $x[n]$ for $n > n_0$, so the system is causal.
4 Convoluted Convolution

a) Show that convolution is commutative. That is, show that 
\[(x * h)[n] = (h * x)[n].\]

**Answer**

\[
(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
= \sum_{m=-\infty}^{\infty} x[n-m]h[m] \quad \text{Let } m = n-k. \\
= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\
= (h * x)[n]
\]

b) Show that \(\delta[n]\) is a convolution identity. That is, show that 
\[(x * \delta)[n] = x[n].\]

**Answer**

Since convolution is commutative, we know that 
\[(x * \delta)[n] = (\delta * x)[n].\]

\[
(\delta * x)[n] = \sum_{k=-\infty}^{\infty} \delta[k]x[n-k]
\]

Since \(\delta[k] = 0\) for all \(k \neq 0\), it follows that 
\[(\delta * x)[n] = \delta[0]x[n] = x[n].\]

Additional Practice:

c) Show that convolution by \(\delta[n-n_0]\) shifts \(x[n]\) by \(n_0\) steps to the right.

**Answer**

Since convolution is commutative \(x[n] * \delta[n-n_0] = \delta[n-n_0] * x[n].\)

\[
\delta[n-n_0] * x[n] = \sum_{k=-\infty}^{\infty} \delta[k-n_0]x[n-k]
\]

Then since \(\delta[k-n_0] = 0\) for all \(k \neq n_0\), it follows that 
\[
\delta[n-n_0] * x[n] = \delta[0]x[n-n_0] = x[n-n_0]
\]

d) Show that convolution is distributive. In other words, show that 
\[(x * (h_1 + h_2))[n] = (x * h_1)[n] + (x * h_2)[n].\]
**Answer**

Since multiplication is distributive, it follows that convolution is distributive

\[
(x * (h_1 + h_2))[n] = \sum_{k=-\infty}^{\infty} x[k](h_1[n - k] + h_2[n - k])
\]

\[
= \sum_{k=-\infty}^{\infty} x[k]h_1[n - k] + \sum_{k=-\infty}^{\infty} x[k]h_2[n - k]
\]

\[
= (x * h_1)[n] + (x * h_2)[n]
\]
5 Mystery System

Consider an LTI system with the following impulse response:

\[ h[n] = \frac{1}{2} (\delta[n] + \delta[n - 1]) \]

(a) Create a sketch of this impulse response.

![Impulse Response Sketch](image)

Answer

(b) What is the output of our system if the input is the unit step \( u[n] \)?

![Output Sketch](image)
Answer

\[ y[n] = (u * h)[n] = \sum_{k=-\infty}^{\infty} u[k] h[n-k] = \sum_{k=0}^{\infty} h[n-k] \]

For \( n < 0 \), \( y[n] = 0 \). When \( n > 0 \),

\[ y[0] = \sum_{k=0}^{\infty} h[-k] = h[0] = 0.5 \]
\[ y[1] = \sum_{k=0}^{\infty} h[1-k] = h[0] + h[1] = 1 \]
\[ y[n] = \sum_{k=0}^{\infty} h[n-k] = h[0] + h[1] + \ldots + h[n] = 1 \text{ for } n > 1. \]

The output \( y[n] \) is shown below.
c) What is the output of our system if our input is \( x[n] = (-1)^n u[n] \)?

![Graph of \( x[n] \) vs. \( n \)]

Answer

\[
y[n] = (u * h)[n] = \sum_{k=-\infty}^{\infty} x[n] h[n - k] = \sum_{k=0}^{\infty} (-1)^k h[n - k]
\]

For \( n < 0 \), \( y[n] = 0 \). When \( n > 0 \),

\[
y[0] = \sum_{k=0}^{\infty} h[-k] = h[0] = 0.5
\]

\[
y[1] = \sum_{k=0}^{\infty} h[1 - k] = h[0] - h[1] = 0
\]

\[
y[2] = \sum_{k=0}^{\infty} h[2 - k] = h[2] - h[1] + h[0] = 0
\]

\[
\vdots
\]

\[
y[n] = 0 \text{ for } n > 0.
\]
d) This system is called the two-point simple moving average (SMA) filter. Based on the previous parts, why do you think it bears this name?

**Answer**

The output of the system at each timestep \( n \) is the average of \( x[n] \) and \( x[n-1] \). To show this formally, we can look at the convolution \( y = x * h \)

\[
y[n] = (x * h)[n] = x * \left( \frac{1}{2} \delta[n] + \frac{1}{2} \delta[n-1] \right)
\]

\[
= \frac{1}{2} x[n] + \frac{1}{2} x[n-1]
\]

This sort of system can be used in areas like technical analysis to gain insight into stock prices and trends (usually these methods would use a longer window than just two days). There are also other variants used like the exponential moving average (EMA) filter.