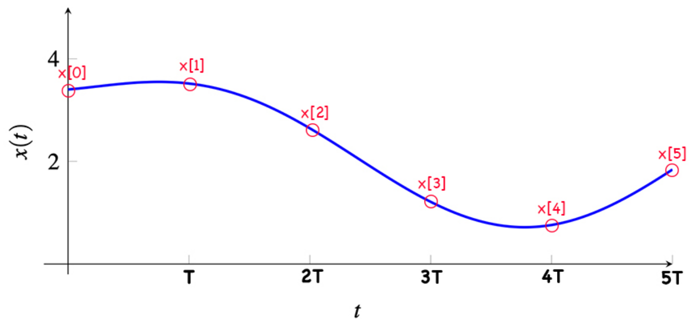
Sampling theorem

Let x be continuous signal bandlimited by frequency ω_{max}. We sample x with a period of T_s.



Given the discrete samples, we can try reconstructing the original signal f through sinc-interpolation where $\Phi(t) = \text{sinc}\left(\frac{t}{T_s}\right)$

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x[n]\Phi(t - nT_s)$$

We define the **sampling frequency** as $\omega_s = \frac{2\pi}{T_s}$. The Sampling Theorem says if $\omega_{max} < \frac{\pi}{T_s}$, or $\omega_s > 2\omega_{max}$, then we are able to recover the original signal, i.e. $x = \hat{x}$.

1 Sampling Theorem basics

Consider the following signal, $x(t)$ defined as,

$$x(t) = \cos(2\pi t). \tag{1}$$

a) Sketch the signal $x(t)$, for $t \in [0, 4]$s.

Answer

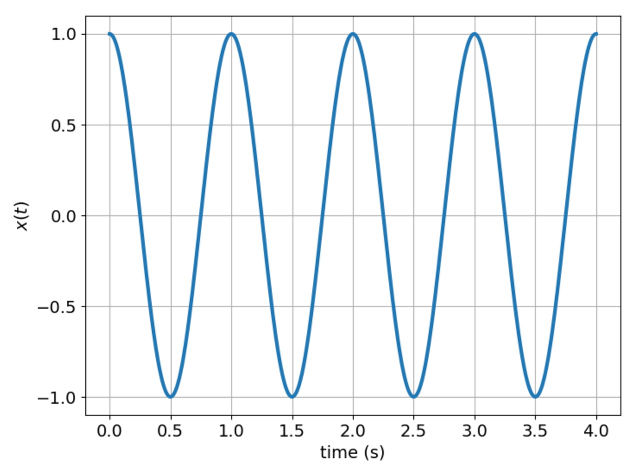


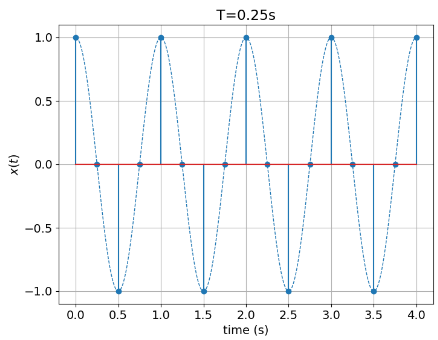
Figure 1

b) Sketch discrete samples of $x(t)$ is the signal is sampled at a period of

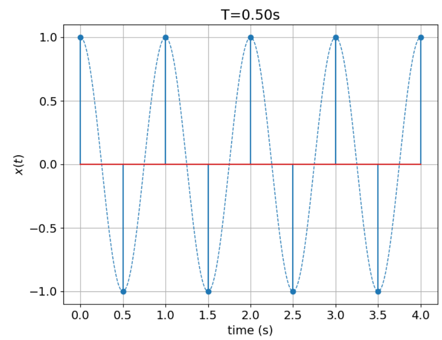
   i) $\frac{1}{4}$s
   ii) $\frac{1}{2}$s
   iii) 1s
   iv) 2s

   How would you reconstruct a continuous signal $\hat{x}(t)$ if you only had the discrete samples for reconstruction?

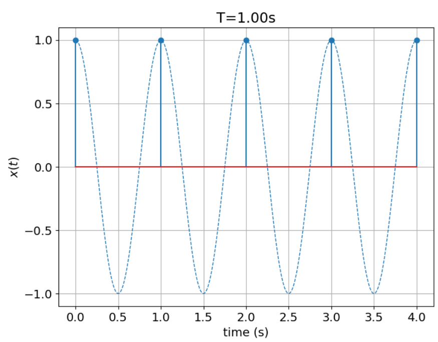
   Answer



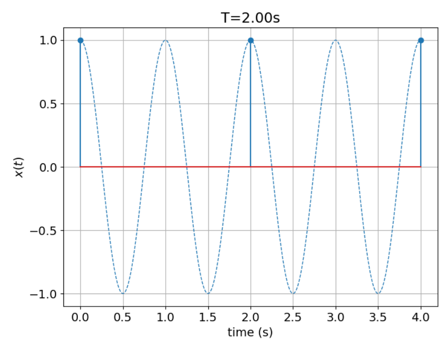
(a) T=$\frac{1}{4}$s.



(b) T=$\frac{1}{2}$s.



(c) T=1s.



(d) T=2s.

Figure 2: Discretizing the continuous time signal $x(t)$ with different sampling periods.

c) What is the maximum frequency, ω_{max}, in radians per second? In Hertz?

   Answer

   $\omega_{\text{max}} = 2\pi$ in radians per second, which is 1 Hertz.

d) If I sample every T seconds, what is the sampling frequency?

Answer

$\omega_s = \frac{2\pi}{T}$.

e) What is the smallest sampling period T that would result in an imperfect reconstruction?

Answer

From the sampling theorem, we know that T has an upperbound of $\frac{\pi}{\omega_{max}}$ for perfect reconstruction. Hence the smallest T for which we cannot reconstruct our signal is,

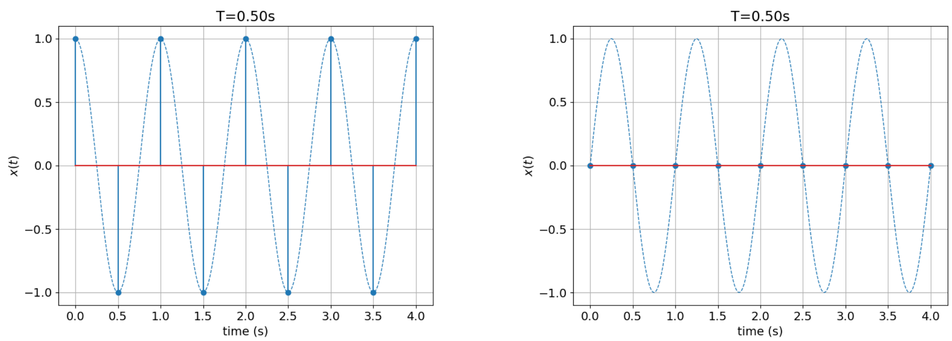
$$T = \frac{\pi}{2\pi} = \frac{1}{2}$$

.

f) Repeat part (b), for the signal

$$y(t) = \sin(2\pi t) \tag{2}$$

Answer



(a) Discretizing $x(t) = \cos(2\pi t)$ at T=$\frac{1}{2}$s. (b) Discretizing $y(t) = \sin(2\pi t)$ at T=$\frac{1}{2}$s.

Figure 3: Comparing discretization of continuous-time signals (a) $x(t)$ and (b) $y(t)$ at the Nyquist frequency.

2 Aliasing

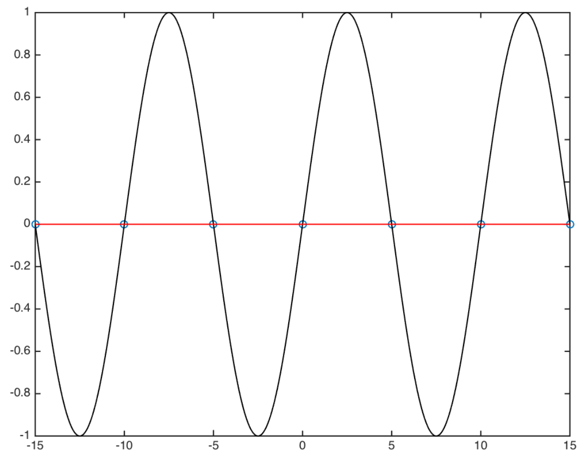
Consider the signal $x(t) = \sin(0.2\pi t)$.

a) At what period T should we sample so that sinc interpolation recovers a function that is identically zero?

Answer

We want to sample such that our resultant discrete time signal is all zeros. To do this, we can sample at $t = 5k$, for integral values of k. Hence, T = 5.

We could also do this graphically by plotting $x(t) = \sin(0.2\pi t)$ and $x(t) = 0$ on the same plot and seeing where they interesect.



b) At what period T can we sample at so that sinc interpolation recovers the function $f(t) = -\sin\left(\frac{\pi}{15}t\right)$?

Answer

T = 7.5

$$\begin{aligned} x[n] &= \sin(0.2\pi nT) && \text{sampling x(t)} \\ &= -\sin(-0.2\pi nT) && \sin(t) \text{ is odd} \\ &= -\sin(-0.2\pi nT + 2\pi n) && \text{For } n \in \mathbb{Z} \text{ since } \sin(t) \text{ is periodic.} \\ &= -\sin\left(\frac{\pi}{15}nT\right) \end{aligned}$$

As a result,

$$\begin{aligned} 2\pi - 0.2\pi T &= \frac{\pi}{15}T \\ T &= 7.5 \end{aligned}$$

As with part (a), we could also do this graphically by plotting $x(t) = \sin(0.2\pi t)$ and $x(t) = -\sin\left(\frac{\pi}{15}t\right)$ on the same plot and looking at the intersection points.