Singular Value Decomposition

The definition

The SVD is a useful way to characterize a matrix. Let A be a matrix from $\mathbb{R}^n$ to $\mathbb{R}^m$ (or $A \in \mathbb{R}^{m\times n}$) of rank r. It can be decomposed into a sum of r rank-1 matrices:

$$A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$$

where

- $\vec{u}_1, \ldots, \vec{u}_r$ are orthonormal vectors in $\mathbb{R}^m$; $\vec{v}_1, \ldots, \vec{v}_r$ are orthonormal vectors in $\mathbb{R}^n$.
- the singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are always real and positive.

We can also re-write the decomposition in matrix form:

$$A = U_1 S V_1^T$$

The properties of U_1, S and V_1 are,

- U_1 is an $[m \times r]$ matrix whose columns consist of $\vec{u}_1, \ldots, \vec{u}_r$. Consequently,

$$U_1^T U_1 = I_{r\times r}$$

- V_1 is an $[n \times r]$ matrix whose columns consist of $\vec{v}_1, \ldots, \vec{v}_r$. Consequently,

$$V_1^T V_1 = I_{r\times r}$$

- U_1 characterizes the column space of A and V_1 characterizes the row space of A.
- S is an $[r \times r]$ matrix whose diagonal entries are the singular values of A arranged in descending order. The singular values are the square roots of the nonzero eigenvalues of $A^T A$ (or, identically, AA^T).

The full matrix form of SVD is

$$A = U \Sigma V^T$$

where $U^T U = I_{m\times m}, V^T V = I_{n\times n}, \Sigma \in \mathbb{R}^{m\times n},$ which contains S and elsewhere zero.

The calculation

We calculate the SVD of matrix A as follows.

(a) Pick $A^T A$ or AA^T.

(b) i. If using $A^T A$, find the eigenvalues λ_i of $A^T A$ and order them, so that $\lambda_1 \geq \cdots \geq \lambda_r > 0$ and $\lambda_{r+1} = \cdots = \lambda_n = 0$.

If using AA^T, find its eigenvalues $\lambda_1, \ldots, \lambda_m$ and order them the same way.

ii. If using $A^T A$, find orthonormal eigenvectors $\vec{v}_i$ such that

$$A^T A\vec{v}_i = \lambda_i \vec{v}_i, \quad i = 1, \dots, r$$

If using AA^T, find orthonormal eigenvectors $\vec{u}_i$ such that

$$AA^T \vec{u}_i = \lambda_i \vec{u}_i, \quad i = 1, \dots, r$$

iii. Set $\sigma_i = \sqrt{\lambda_i}$.

If using $A^T A$, obtain $\vec{u}_i$ from $\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i, \quad i = 1, \dots, r.$

If using AA^T, obtain $\vec{v}_i$ from $\vec{v}_i = \frac{1}{\sigma_i} A^T \vec{u}_i, \quad i = 1, \dots, r.$

(c) This is not in scope but if you want to completely construct the U or V matrix, complete the basis (or columns of the appropriate matrix) using Gram-Schmidt. Remember to normalize afterwards.

The full matrix form of SVD is taken to better understand the matrix A in terms of the 3 nice matrices U, Σ, V. Often, we do not completely construct the U and V matrices.

1 SVD and Fundamental Subspaces

Define the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

a) Find the SVD of A (compact form is fine).

Answer

First, compute $A^T A = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$. The eigenvalues of $A^T A$ are 18 and 0, with corresponding unit eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

Therefore, A has one singular vector $\sqrt{18} = 3\sqrt{2}$
We obtain

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

and A can be decomposed as

$$A = 3\sqrt{2} \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

b) Find the rank of A.

Answer

A has 1 nonzero singular value. So A has rank 1.

c) Find a basis for the kernel (or nullspace) of A.

Answer

$$\ker(A) = \operatorname{span}\left\{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right\}$$

d) Find a basis for the range (or columnspace) of A.

Answer

$$\operatorname{range}(A) = \operatorname{span}\left\{\begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}\right\}$$

e) Repeat parts (a) - (d) for A^T instead. What are the relationships between the answers for A and the answers for A^T?

Answer

$$A^T = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

$$\lambda = 18, 0$$

$$\vec{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

$$\vec{v}_1 = \frac{1}{\sigma_1} A^T \vec{u}_1 = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

$$A = 3\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/3 & -2/3 & 2/3 \end{bmatrix}$$

At this point, we already know the rank is 1. The column space is also formed by the $\vec{u}$ vector

$$\operatorname{range}(A) = \operatorname{span}\left\{\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}\right\}$$

Two vectors in the nullspace are

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

Note, if we had just noticed

$$A^T = (U\Sigma V^T)^T = V\Sigma^T U^T$$

We could've skippepd many steps for SVD calculation.

2 Eigenvalue Decomposition and Singular Value Decomposition

We define Eigenvalue Decomposition as follows:

If a matrix $A \in \mathbb{R}^{n \times n}$ has n linearly independent eigenvectors $\vec{p}_1, \ldots, \vec{p}_n$ with eigenvalues $\lambda_i, \ldots, \lambda_n$, then we can write:

$$A = P\Lambda P^{-1}$$

Where columns of P consist of $\vec{p}_1, \ldots, \vec{p}_n$, and Λ is a diagonal matrix with diagonal entries $\lambda_i, \ldots, \lambda_n$.

Consider a matrix $A \in \mathbb{S}^n$, that is, $A = A^T \in \mathbb{R}^{n \times n}$. This is a symmetric matrix and has orthorgonal eigenvectors. Therefore its eigenvalue decomposition can be written as,

$$A = P\Lambda P^T$$

a) First, assume $\lambda_i \geq 0, \forall i$. Find a SVD of A.

Answer

Observe that,

$$A^T A = P\Lambda^2 P^T$$

This means that,

$$\sigma_i = \lambda_i \text{ and } V = P$$

We have,

$$Av_i = \lambda_i v_i = \sigma_i v_i$$

Plugging into our SVD condition $Av_i = \sigma_i u_i$:

$$\sigma_i v_i = \sigma_i u_i$$

This means that,

$$U = V = P$$

Therefore, in this case, the eigenvalue decomposition is the same as the singular value decompositions.

b) Let one particular eigenvalue λ_j be negative, with the associated eigenvector being p_j. Succinctly,

$$Ap_j = \lambda_j p_j \text{ with } \lambda_j < 0$$

We are still assuming that,

$$A = P\Lambda P^T$$

a) What is the singular value σ_j associated to λ_j?

b) What is the relationship between the left singular vector u_j, the right singular vector v_j and the eigenvector p_j?

Answer

a)

$$\sigma_j = |\lambda_j|$$

b) Either,

$$u_j = p_j \text{ and } v_j = -p_j$$

or,

$$u_j = -p_j \text{ and } v_j = p_j$$

This is because the diagonal entries of Σ MUST be non-negative.