Discussion 7A

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Q1.

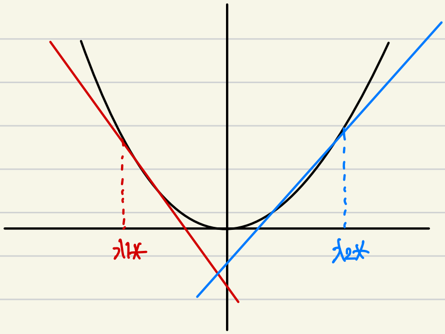
Recall : Taylor Expansion

$$f(x) = f(x^*) + \frac{f'(x^*)(x-x^*)}{1!} + \frac{f''(x^*)(x-x^*)^2}{2!} + \cdots$$

"operating point"

→ point which we want to approximate/linearize around

→ something that we can choose



Given : $f(x) = x^3 - 3x^2$

a) $f(x) \simeq f(x^*) + f'(x^*)(x-x^*)$

$= f(x^*) + (3x^2 - 6x)\Big|_{x=x^*} (x - x^*)$

b) Given equilibrium point $x^* = 1.5$

$f(x) \simeq f(1.5) + (3\cdot 1.5^2 - 6\cdot 1.5)(x-1.5)$

$= -3.375 - 2.25(x-1.5)$ ← linear in x

$\hat{f}(x=1.7)$ VS $\hat{f}(x=2.5)$

$\begin{cases} \hat{f}(x=1.7) = -3.885 \\ f(x=1.7) = -3.757 \end{cases}$

$\begin{cases} \hat{f}(x=2.5) = -5.625 \\ f(x=2.5) = -3.725 \end{cases}$ pretty bad!

closer to operating pt x^* → fairly good approx.

Given function in two variables,

$$f(x,y) \approx f(x^*,y^*) + \left.\frac{\partial f}{\partial x}\right|_{x^*,y^*}(x-x^*) + \left.\frac{\partial f}{\partial y}\right|_{x^*,y^*}(y-y^*)$$

operating points

Partial Derivative → consider other variables as constants

→ ex) $f(x,y) = 4x^3y^2$

$$\frac{\partial f}{\partial x} = 4\cdot(3x^2)y^2 = 12x^2y^2$$

$$\frac{\partial f}{\partial y} = 4x^3(2y) = 8x^3y$$

Given $f(x,y) = x^2y$

→ c) $\frac{\partial f}{\partial x} = 2xy \quad \frac{\partial f}{\partial y} = x^2$

d) $f(x,y) \approx f(x^*,y^*) + 2x^*y^*(x-x^*) + x^{*2}(y-y^*)$

e) Given $(x^*,y^*) = (2,3)$

$$f(x,y) \approx 12 + 2\cdot2\cdot3(x-2) + 4(y-3) = 12 + 12(x-2) + 4(y-3)$$

$f(x^*,y^*) = f(2,3) = 2^2\cdot3 = 12$

(continued)

e) $f(x,y) \simeq 12+12(x-2)+4(y-3)$

→ Approximation at $(2+\delta, 3+\delta)$

$f(2+\delta, 3+\delta) \simeq 12+12(2+\delta-2)+4(3+\delta-3) = 12+16\delta$

→ Actual value at $(2+\delta, 3+\delta)$

$f(x,y) = x^2 y$

$= (2+\delta)^2(3+\delta) = 12+16\delta+7\delta^2+\delta^3$

distance from operating point

$\Rightarrow$ Error $= |\delta^3+7\delta^2|$ ← error gets exponentially small as δ gets smaller

at $\delta = 0.01$, error is $(0.01)^3+7(0.01)^2 = 0.000701$

Extend this further: $f(\vec{x}) \rightarrow$ function that takes in a vector $\in \mathbb{R}^n$

i.e. $x_1, x_2, \dots, x_n$ as your input

$$\vec{x} \in \mathbb{R}^n \rightarrow \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\Rightarrow f(\vec{x}) \approx f(\vec{x}_*) + \left.\frac{\partial f}{\partial x_1}\right|_{x_1 = x_{1*}} (x_1 - x_{1*}) + \left.\frac{\partial f}{\partial x_2}\right|_{x_2 = x_{2*}} (x_2 - x_{2*}) + \dots + \left.\frac{\partial f}{\partial x_n}\right|_{x_n = x_{n*}} (x_n - x_{n*})$$

$$= f(\vec{x}_*) + \sum_{i=1}^{n} \left.\frac{\partial f}{\partial x_i}\right|_{x_i = x_{i*}} (x_i - x_{i*})$$

$$= f(\vec{x}_*) + \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \dots & \frac{\partial f}{\partial x_n} \end{bmatrix}}_{J_{\vec{x}} f} \begin{bmatrix} x_1 - x_{1*} \\ x_2 - x_{2*} \\ \vdots \\ x_n - x_{n*} \end{bmatrix}$$

$$= f(\vec{x}_*) + J_{\vec{x}} f (\vec{x} - \vec{x}_*)$$

$$\Rightarrow f(\underbrace{\vec{x}}_{\mathbb{R}^n}, \underbrace{\vec{y}}_{\mathbb{R}^k}) \approx f(\vec{x}_*, \vec{y}_*) + \underbrace{J_{\vec{x}} f (\vec{x} - \vec{x}_*)}_{n \text{ terms}} + \underbrace{J_{\vec{y}} f (\vec{y} - \vec{y}_*)}_{k \text{ terms}}$$

f) Given: $f(\vec{x}, \vec{y}) = \vec{x}^T \vec{y} = \cancel{x_1 y_1} + x_2 y_2 + \dots + \cancel{x_n y_n}$

$$J_{\vec{x}} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \dots & \frac{\partial f}{\partial x_n} \end{bmatrix} = \begin{bmatrix} y_1 & y_2 & \dots & y_n \end{bmatrix} = \vec{y}^T$$

$$J_{\vec{y}} f = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \dots & \frac{\partial f}{\partial y_n} \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} = \vec{x}^T$$

g) From part (f) : $J_{\vec{x}} f = \vec{y}^T$, $J_{\vec{y}} f = \vec{x}^T$

Given : $f(\vec{x}, \vec{y}) = \vec{x}^T \vec{y}$, $\vec{x}_* = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{y}_* = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$f(\vec{x}, \vec{y}) \approx f(\vec{x}_*, \vec{y}_*) + \vec{y}_*^T(\vec{x} - \vec{x}_*) + \vec{x}_*^T(\vec{y} - \vec{y}_*)$$

$$= \vec{x}_*^T \vec{y}_* + \vec{y}_*^T(\vec{x} - \vec{x}_*) + \vec{x}_*^T(\vec{y} - \vec{y}_*)$$

$$= 3 + \begin{bmatrix} -1 \\ 2 \end{bmatrix}^T \left(\vec{x} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) + \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T \left(\vec{y} - \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$$