Discussion 6C

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EECS 16B Su22

Q1. Complex inner product

- Recall: $\langle \vec{x}, \vec{y} \rangle = \vec{y}^T \vec{x} = \vec{x}^T \vec{y} = \langle \vec{x}, \vec{y} \rangle$ if $\vec{x}, \vec{y} \in \mathbb{R}^n$

  $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

  ↳ but NOT hold for complex case

→ $\vec{u}, \vec{v} \in \mathbb{C}^n$

$\langle \vec{u}, \vec{v} \rangle = \vec{v}^{*} \vec{u}$ conjugate transpose

$= \sum_{i=1}^{n} \overline{v_i} \cdot u_i$

plug into projection formula given by the question

a) WTS: $\text{proj}_{\vec{u}}(\alpha \vec{u}) = \alpha \vec{u}$

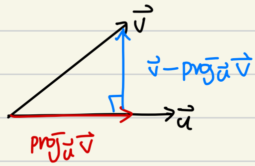
$$\text{proj}_{\vec{u}}(\alpha \vec{u}) = \frac{\langle \alpha \vec{u}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle} \cdot \vec{u}$$

$$= \frac{\vec{u}^{*}(\alpha \vec{u})}{\vec{u}^{*} \vec{u}} \cdot \vec{u}$$

$$= \frac{\alpha \cancel{\vec{u}^{*} \vec{u}}}{\cancel{\vec{u}^{*} \vec{u}}} \vec{u} = \alpha \vec{u} \checkmark$$

b) orthogonality principle : $\langle \vec{u}, \vec{v} \rangle = \vec{v}^* \vec{u} = 0$

WTS: $\langle \vec{u}, \vec{v} - proj_{\vec{u}} \vec{v} \rangle = 0$



let's show this mathematically!

$\rightarrow \langle \vec{u}, \vec{v} - proj_{\vec{u}}(\vec{v}) \rangle$

$= (\vec{v} - proj_{\vec{u}} \vec{v})^* \vec{u}$

$= \left(\vec{v} - \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle} \vec{u} \right)^* \vec{u}$

$(A-B)^T = A^T - B^T$
$(A-B)^* = A^* - B^*$

$= \vec{v}^* \vec{u} - \left(\frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle} \cdot \vec{u} \right)^* \vec{u}$

$= \vec{v}^* \vec{u} - \left(\frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle} \right)^* \vec{u}^* \vec{u}$

squared norm $\|\vec{u}\|^2$

$= \vec{v}^* \vec{u} - \frac{\langle \vec{v}, \vec{u} \rangle^*}{\langle \vec{u}, \vec{u} \rangle^*} \cdot \vec{u}^* \vec{u}$ $\quad \langle \vec{u}, \vec{u} \rangle \in \mathbb{R}$

$\therefore \langle \vec{u}, \vec{u} \rangle^* = \langle \vec{u}, \vec{u} \rangle$ (no complex component!)

$= \vec{v}^* \vec{u} - \langle \vec{v}, \vec{u} \rangle^* \cdot \cancel{\frac{\vec{u}^* \vec{u}}{\langle \vec{u}, \vec{u} \rangle}}^{\,1}$

$= \vec{v}^* \vec{u} - \langle \vec{v}, \vec{u} \rangle^* = \vec{v}^* \vec{u} - (\vec{u}^* \vec{v})^* = \vec{v}^* \vec{u} - \vec{v}^* \vec{u} = 0$ ✓

$(AB)^* = B^* A^*$

Note: Generally for $\vec{u}, \vec{v} \in \mathbb{C}^n$, $\langle \vec{u}, \vec{v} \rangle \neq \langle \vec{v}, \vec{u} \rangle$

$\rightarrow$ Correct: $\langle \vec{u}, \vec{v} \rangle = (\langle \vec{v}, \vec{u} \rangle)^*$

But orthogonal case doesn't matter b/c they end up being zero i.e.

$$\langle \vec{v} - \text{proj}_{\vec{u}}(\vec{v}), \vec{u} \rangle = (\langle \vec{u}, \vec{v} - \text{proj}_{\vec{u}}(\vec{v}) \rangle)^* = (0)^* = 0$$

c) $\text{proj}_{col(A)}(\vec{u}) = A(A^*A)^{-1}A^*\vec{u}$ formula given by the question

WTS: If $\vec{u} = A\vec{x}$, $\vec{x} \in \mathbb{C}$, then $\text{proj}_{col(A)}(\vec{u}) = \vec{u}$

$$\text{proj}_{col(A)}(\vec{u}) = A(A^*A)^{-1}A^*\vec{u}$$

$$= A(A^*A)^{-1}A^*(A\vec{x})$$

$$= \underbrace{A(A^*A)^{-1}(A^*A)}_{I}\vec{x} = A\vec{x} = \vec{u}$$

d) Show that $\langle \vec{u} - \text{proj}_{col(A)}(\vec{u}), \vec{b} \rangle = 0$ if $\vec{b} \in col(A)$

$$\langle \vec{u} - \text{proj}_{col(A)}(\vec{u}), \vec{b} \rangle = \vec{b}^*(\vec{u} - \text{proj}_{col(A)}(\vec{u}))$$

$$= (A\vec{x})^*(\vec{u} - \text{proj}_{col(A)}(\vec{u}))$$

$$= \vec{x}^*A^*(\vec{u} - \text{proj}_{col(A)}(\vec{u}))$$

$$= \vec{x}^*A^*\vec{u} - \vec{x}^*A^*\underbrace{\text{proj}_{col(A)}(\vec{u})}_{= A(A^*A)^{-1}A^*\vec{u}} = \vec{x}^*A^*\vec{u} - \vec{x}^*\underbrace{A^*A(A^*A)^{-1}}_{I}A^*\vec{u}$$

$$= \vec{x}^*A^*\vec{u} - \vec{x}^*A^*\vec{u} = 0 \checkmark$$

e) A is orthonormal iff $\begin{cases} \langle \vec{a_i}, \vec{a_j} \rangle = \vec{a_j}^* \vec{a_i} = 0 \\ \|\vec{a_i}\| = \sqrt{\langle \vec{a_i}, \vec{a_i} \rangle} = \sqrt{\vec{a_i}^* \vec{a_i}} = 1 \end{cases}$

$\rightarrow A^*A = I$

$A = [\vec{a_1} \cdots \vec{a_n}]$

$A^* = \begin{bmatrix} \vec{a_1}^* \\ \vdots \\ \vec{a_n}^* \end{bmatrix} \longleftrightarrow A^T = \begin{bmatrix} \vec{a_1}^T \\ \vdots \\ \vec{a_n}^T \end{bmatrix}$

$\therefore A^*A = \begin{bmatrix} \vec{a_1}^* \\ \vdots \\ \vec{a_n}^* \end{bmatrix} [\vec{a_1} \cdots \vec{a_n}]$

$\langle \vec{a_1}, \vec{a_1} \rangle = \underbrace{\|\vec{a_1}\|^2}_{1} \in \mathbb{R}$

$= \begin{bmatrix} \vec{a_1}^* \vec{a_1} & \cdots & \vec{a_1}^* \vec{a_n} \\ \vdots & \ddots & \\ \vec{a_n}^* \vec{a_1} & & \vec{a_n}^* \vec{a_n} \end{bmatrix}$

$= \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I_{n \times n}$ ✓

Plugging into (c),

$\text{proj}_{col(A)} \vec{u} = A(\underbrace{A^*A}_{I})^{-1} A^* \vec{u} = AA^* \vec{u}$