Discussion 5C

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Q1.

Orthonormality

- For a $n \times n$ orthonormal square matrix U, the following properties hold:

$$\begin{cases} U^T U = U U^T = I_{n \times n} \\ U^T = U^{-1} \end{cases}$$

a) $\vec{y_1} = U \vec{x_1}$

$\vec{y_2} = U \vec{x_2}$

$$\langle \vec{y_1}, \vec{y_2} \rangle = \vec{y_2}^T \vec{y_1}$$

$$= (U\vec{x_2})^T U \vec{x_1}$$

$$= \vec{x_2}^T \underbrace{U^T U}_{I} \vec{x_1} = \vec{x_2}^T \vec{x_1} = \langle \vec{x_1}, \vec{x_2} \rangle$$

$\Rightarrow$ Inner product is preserved in the new basis

b) Recall: $\|\vec{x}\|^2 = \vec{x}^T \vec{x} = \langle \vec{x}, \vec{x} \rangle$

↳ Why?

$$\underset{\in \mathbb{R}^n}{\vec{v}} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \qquad \|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

$$\left.\begin{matrix} \|\vec{v}\|^2 = v_1^2 + v_2^2 + \cdots + v_n^2 \\ \vec{v}^T \vec{v} = [v_1 \cdots v_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = v_1^2 + v_2^2 + \cdots + v_n^2 \end{matrix}\right\} \|\vec{v}\|^2 = \vec{v}^T \vec{v}$$

$$\|\vec{y_1}\|^2 = \vec{y_1}^T \vec{y_1} = \langle \vec{y_1}, \vec{y_1} \rangle = \langle U\vec{x_1}, U\vec{x_1} \rangle = \vec{x_1}^T \overbrace{U^T U}^{I} \vec{x_1} = \vec{x_1}^T \vec{x_1} = \|\vec{x_1}\|^2$$

$\Rightarrow \|\vec{y_1}\|^2 = \|\vec{x_1}\|^2$

$\|\vec{y_2}\|^2 = \|\vec{x_2}\|^2$

c) $y_i = \vec{a}^T \vec{x_i}$ $\rightarrow$ $y_1 = \vec{a}^T \vec{x_1}$

$$y_2 = \vec{a}^T \vec{x_2}$$

$$\vdots$$

$$y_m = \vec{a}^T \vec{x_m}$$

Try to write these equations in matrix-vector form

Goal: $\vec{y} = X\vec{a}$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \vec{a}^T \begin{bmatrix} \vec{x_1} \\ \vec{x_2} \\ \vdots \\ \vec{x_m} \end{bmatrix} = \underbrace{\begin{bmatrix} \vec{x_1}^T \\ \vec{x_2}^T \\ \vdots \\ \vec{x_m}^T \end{bmatrix}}_{X} \vec{a}$$

can change the order of dot product

$\Rightarrow$ To estimate for $\vec{a}$, use least squares:

$$\hat{\vec{a}} = (X^T X)^{-1} X^T \vec{y} \quad \text{where } X = \begin{bmatrix} \vec{x_1}^T \\ \vdots \\ \vec{x_m}^T \end{bmatrix}$$

orthonormal

d) $X = MV^T$ plug this into LS solution from part (c)

$\Rightarrow \hat{\vec{a}} = \left((MV^T)^T (MV^T) \right) (MV^T)^T \vec{y}$

Note: $(AB)^T = B^T A^T$

$(AB)^{-1} = B^{-1} A^{-1}$

$$= (VM^T M V^T)^{-1} V M^T \vec{y}$$

$$= (V^T)^{-1} (M^T M)^{-1} (V)^{-1} V M^T \vec{y}$$

$$= (V^{-1})^{-1} (M^T M)^{-1} \underbrace{V^T V}_{I} M^T \vec{y} = V (M^T M)^{-1} M^T \vec{y}$$

e) $X = U\Sigma V^T$ [SVD of a matrix]

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \\ \hline & 0 & \end{bmatrix}$$

In part (d), $X = \underline{M}V^T$

$\rightarrow$ Now in part (e), $X = \underline{U\Sigma}V^T$ $\rightarrow$ plug in $M = U\Sigma$ into sol'n from (d)

From part (d), $\hat{a} = V(M^TM)^{-1}M^T\vec{y}$

$$\Rightarrow \hat{a} = V((U\Sigma)^T(U\Sigma))^{-1}(U\Sigma)^T\vec{y}$$

$$= V(\Sigma^T\underbrace{U^TU}_{I}\Sigma)^{-1}\Sigma^TU^T\vec{y}$$

$$= V(\Sigma^T\Sigma)^{-1}\Sigma^TU^T\vec{y} \qquad \Sigma^T = \left[\begin{array}{ccc|c} \sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_n & \end{array}\right]$$

$$= V\left(\underbrace{\left[\begin{array}{ccc|c} \sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_n & \end{array}\right]}_{n\times m}\underbrace{\begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \\ \hline & 0 & \end{bmatrix}}_{m\times n}\right)^{-1}\Sigma^TU^Ty$$

$$= V\left(\begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_n^2 \end{bmatrix}\right)^{-1}\Sigma^TU^Ty$$

$$= V\underbrace{\begin{bmatrix} 1/\sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & 1/\sigma_n^2 \end{bmatrix}}_{n\times n}\underbrace{\left[\begin{array}{ccc|c} \sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_n & \end{array}\right]}_{n\times m}U^Ty = V\left[\begin{array}{ccc|c} 1/\sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & 1/\sigma_n & \end{array}\right]U^Ty$$