Discussion 4D

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Q1.

a) $\vec{x}[i+1] = \underbrace{\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}}_{A} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \vec{u}[i] + \vec{w}[i]$

Stability: All $|\lambda_i| < 1$

$\downarrow$

λ_i's = eigenvalues of A

Recall: $\det(A - \lambda I) = 0$

$$\underbrace{\det\begin{pmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{pmatrix}}_{\Downarrow} = 0$$

$$= -\lambda(-1-\lambda) - 2 \cdot 1 = \lambda + \lambda^2 - 2 = (\lambda+2)(\lambda-1) = 0$$

$\therefore \lambda_1 = -2, \ \lambda_2 = 1 \Rightarrow$ NOT stable

both $|\lambda_i| \geq 1$

b) $u[i] = [f_1 \ f_2] \vec{x}[i]$

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left([f_1 \ f_2] \vec{x}[i] \right)$$

$$= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} f_1 & f_2 \\ 0 & 0 \end{bmatrix} \vec{x}[i]$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \\ 0 & 0 \end{bmatrix} \right) \vec{x}[i]$$

$$= \underbrace{\begin{bmatrix} f_1 & 1+f_2 \\ 2 & -1 \end{bmatrix}}_{= A_{CL}} \vec{x}[i]$$

c) $\boxed{\lambda_1 = -\frac{1}{2},\ \lambda_2 = \frac{1}{2}}$ $\rightarrow$ $\det(A_{CL} - \lambda I) = 0$

$$\parallel$$

$$(\lambda + \tfrac{1}{2})(\lambda - \tfrac{1}{2}) = 0$$

From part (b),

$$\vec{x}[i+1] = \underbrace{\begin{bmatrix} f_1 & 1+f_2 \\ 2 & -1 \end{bmatrix}}_{A_{CL}} \vec{x}[i]$$

$$\Rightarrow \det(A_{CL} - \lambda I) = 0$$

$$\det\begin{pmatrix} f_1-\lambda & 1+f_2 \\ 2 & -1-\lambda \end{pmatrix} = 0$$

$$(f_1-\lambda)(-1-\lambda) - 2\cdot(1+f_2) = -f_1 + (1-f_1)\lambda + \lambda^2 - 2 - 2f_2 = 0$$

$$\rightarrow \lambda^2 + (1-f_1)\lambda - f_1 - 2f_2 - 2 = 0$$

$$\Updownarrow$$

$$(\lambda + \tfrac{1}{2})(\lambda - \tfrac{1}{2}) = \lambda^2 - \frac{1}{4} = 0$$

$$\Rightarrow \begin{cases} 1 - f_1 = 0 \longrightarrow f_1 = 1 \\ -f_1 - 2f_2 - 2 = -\frac{1}{4} \end{cases}$$

$\hookrightarrow -1 - 2f_2 - 2 = -\frac{1}{4}$

$-2f_2 = \frac{11}{4}$, $\therefore f_2 = \frac{-11}{8}$

d) W/ feedback from (c), $\lambda_1 = -\frac{1}{2}$, $\lambda_2 = \frac{1}{2}$ $\Rightarrow$ all / both $|\lambda_i| < 1$ $\Rightarrow$ stable

e) $\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[i]$

$u[i] = [f_1 \ f_2] \vec{x}[i]$

$\Rightarrow \vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left([f_1 \ f_2] \vec{x}[i] \right)$

$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} [f_1 \ f_2] \right) \vec{x}[i]$

$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} f_1 & f_2 \\ f_1 & f_2 \end{bmatrix} \right) \vec{x}[i] = \underbrace{\begin{bmatrix} f_1 & f_2+1 \\ f_1+2 & f_2-1 \end{bmatrix}}_{A_{CL}} \vec{x}[i]$

$\Rightarrow$ Find eigenvalues

$\det(A_{CL} - \lambda I) = 0$

$\det \begin{pmatrix} f_1 - \lambda & 1+f_2 \\ 2+f_1 & -1+f_2-\lambda \end{pmatrix} = 0$

$(f_1-\lambda)(-1+f_2-\lambda) - (1+f_2)(2+f_1) = 0$

$f_1f_2 - f_1 - f_1\lambda - \lambda f_2 + \lambda + \lambda^2 - f_1f_2 - f_1 - 2f_2 - 2 = 0$

$\lambda^2 + (1-f_1-f_2)\lambda - 2(1+f_1+f_2) = 0$

$\Rightarrow (\lambda+2)(\lambda - (1+f_1+f_2)) = 0$

$\lambda_1 = -2 \quad |\lambda_i| \geq 1 \Rightarrow$ unstable, regardless of $f_1 + f_2$

Q2. $\vec{x}[i+1] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i]$

$\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

a) Recall: Given $\vec{x}[i+1] = \underbrace{A}_{n\times n}\vec{x}[i] + \vec{b}u[i]$

$C = [\, A^{n-1}\vec{b} \quad A^{n-2}\vec{b} \; \cdots \; \vec{b} \,] \rightarrow$ lin. indep $\Rightarrow$ controllable

$A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

$A\vec{b} = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$A^2\vec{b} = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

$\therefore \; C = [\, A^{2}\vec{b} \;\; A\vec{b} \;\; \vec{b} \,] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

rank(C) = 2 , not controllable

b) $\vec{x}[l] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for some l

$$\vec{x}[i+1] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i], \quad \vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x}[i] = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x_1[i-1] \\ x_2[i-1] \\ x_3[i-1] \end{bmatrix}}_{\vec{x}[i-1]} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u[i-1]$$

$$= \begin{bmatrix} 2x_1[i-1] \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2u[i-1] \end{bmatrix}$$

$$\vec{x}[i] = \begin{bmatrix} 2x_1[i-1] \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] \end{bmatrix} = \begin{bmatrix} 2^i \\ -3x_1[i-1] + x_3[i-1] \\ x_2[i-1] + 2u[i-1] \end{bmatrix}$$

Goal: $\vec{x}[l] = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$

$$\vec{x}[1] = \begin{bmatrix} 2 \\ -3+0 \\ 0 + 2u[0] \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 2u[0] \end{bmatrix}$$

we cannot reach $\vec{x}[l]$ w/ $\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\vec{x}[2] = \begin{bmatrix} 2\cdot 2 \\ -6 + 2u[0] \\ -3 + 2u[1] \end{bmatrix} = \begin{bmatrix} 4 \\ -6 + 2u[0] \\ -3 + 2u[0] \end{bmatrix}$$

$$\vec{x}[3] = \begin{bmatrix} 8 \\ * \\ * \end{bmatrix}$$

⋮

d) $\vec{x}[2] = \begin{bmatrix} 2^2 \\ -6 + 2u[0] \\ -3 + 2u[1] \end{bmatrix} = \begin{bmatrix} 4 \\ -6 + 2u[0] \\ -3 + 2u[1] \end{bmatrix}$

$= \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -6 + 2u[0] \\ -3 + 2u[1] \end{bmatrix}}$

$= \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$