Discussion 2C

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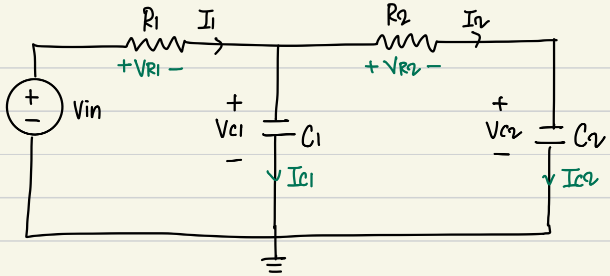
EECS 16B Su22

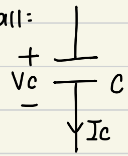
Q1.



a) $C_1 = 1\mu F = 1 \cdot 10^{-6} F$ $\qquad$ $R_1 = \frac{1}{3} M\Omega = \frac{1}{3} \cdot 10^6 \Omega$

$C_2 = \frac{1}{3} \mu F = \frac{1}{3} \cdot 10^{-6} F$ $\qquad$ $R_2 = \frac{1}{2} M\Omega = \frac{1}{2} \cdot 10^6 \Omega$

Recall:

→ 

$I_c = C \cdot \frac{dV_c}{dt}$

① NVA:

$V_{R1} + V_{C1} - V_{in} = 0$ (left) $\rightarrow V_{C1} = V_{in} - V_{R1} = V_{in} - I_1 R_1$, $I_1 = \frac{V_{in} - V_{C1}}{R_1}$ (1)

$V_{R2} + V_{C2} - V_{C1} = 0$ (right) $\rightarrow V_{C2} = V_{C1} - V_{R2} = V_{C1} - I_2 R_2$, $I_2 = \frac{V_{C1} - V_{C2}}{R_2}$ (2)

② KCL:

$I_{C1} = I_1 - I_2$

$I_{C2} = I_2$

③ Element Equation (capacitor)

$I_{C2} = C_2 \cdot \frac{dV_{C2}}{dt} = I_2 \underset{(2)}{\Rightarrow} C_2 \frac{dV_{C2}}{dt} = \frac{V_{C2} - V_{C1}}{R_2}$, $\frac{dV_{C2}}{dt} = \frac{V_{C2} - V_{C1}}{R_2 C_2}$

$I_{C1} = C_1 \cdot \frac{dV_{C1}}{dt} = I_1 - I_2 \underset{(1),(2)}{\Rightarrow} C_1 \cdot \frac{dV_{C1}}{dt} = \frac{V_{in}}{R_1} - \frac{V_{C1}}{R_1} - \frac{V_{C1}}{R_2} + \frac{V_{C2}}{R_2}$

$= -\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) V_{C1} + \frac{V_{C2}}{R_2 C_1} + \frac{V_{in}}{R_1 C_1}$

a) (continued)

In summary,

$$\begin{cases} \dfrac{dV_{C1}}{dt} = -\left(\dfrac{1}{R_1C_1} + \dfrac{1}{R_2C_1}\right)V_{C1} + \dfrac{V_{C2}}{R_2C_1} + \dfrac{V_{in}}{R_1C_1} \\ \dfrac{dV_{C2}}{dt} = \dfrac{V_{C1}}{R_2C_2} - \dfrac{V_{C2}}{R_2C_2} \end{cases}$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} V_{C1}(t) \\ V_{C2}(t) \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} \begin{bmatrix} V_{C1}(t) \\ V_{C2}(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix} V_{in}$$

Plug in the given values of
$$\begin{cases} C_1 = 1 \cdot 10^{-6} \\ C_2 = \frac{1}{3} \cdot 10^{-6} \\ R_1 = \frac{1}{3} \cdot 10^{6} \\ R_2 = \frac{1}{2} \cdot 10^{6} \end{cases}$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} V_{C1}(t) \\ V_{C2}(t) \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} V_{C1}(t) \\ V_{C2}(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} V_{in}(t)$$

b) $V_{C1}(t) = z_1(t)$, $V_{C2}(t) = z_2(t)$, $V_{in} = 0$

$z_1(0) = 7$, $z_2(0) = 7$

$$\frac{d}{dt}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \cancel{\begin{bmatrix} 3 \\ 0 \end{bmatrix} \underbrace{V_{in}(t)}_{= 0}}$$

$$\frac{d}{dt}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}}_{\vec{z}(t)}$$

Initial condition : $\vec{z}(0) = \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$

c) $A = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \rightarrow \lambda_1, \lambda_2$?

$\det(A - \lambda I) = 0$

$$\det\left(\begin{bmatrix} -5-\lambda & 2 \\ 6 & -6-\lambda \end{bmatrix}\right) = (-5-\lambda)(-6-\lambda) - 2\cdot 6$$

$$= 30 + 11\lambda + \lambda^2 - 12$$

$$= \lambda^2 + 11\lambda + 18 = (\lambda+9)(\lambda+2) = 0 \rightarrow \lambda_1 = -9,\ \lambda_2 = -2$$

① $\lambda_1 = -9 \rightarrow$ plug back to $A - \lambda I$

$$\begin{bmatrix} -5+9 & 2 \\ 6 & -6+9 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$$

$$\begin{cases} 4v_1 + 2v_2 = 0 \\ 6v_1 + 3v_2 = 0 \end{cases}$$

Find $\vec{v_1}$ s.t.

$$\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}\underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\vec{v_1}} = \begin{bmatrix} 4v_1 + 2v_2 \\ 6v_1 + 3v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = -\frac{1}{2}v_2$$

scalar

$$\vec{v_1} = \alpha \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

② $\lambda_2 = -2$

$$A - \lambda_2 I = \begin{bmatrix} -5+2 & 2 \\ 6 & -6+2 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 6 & -4 \end{bmatrix}$$

Find $\vec{v_2} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ s.t.

$$\begin{bmatrix} -3 & 2 \\ 6 & -4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -3v_1 + 2v_2 \\ 6v_1 - 4v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 3v_1 = 2v_2$, $\quad \vec{v_2} = \beta \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

d) From c) : $V = [\vec{v_1}\ \vec{v_2}] = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$

Recall :

$$\begin{tikzcd} \vec{z}(t) \arrow[rr, bend left, "A"] \arrow[d, bend right, "V^{-1}"'] & & \frac{d}{dt}\vec{z}(t) \arrow[d, bend right, "V^{-1}"'] \\ \vec{y}(t) \arrow[u, bend right, "V"'] & & \frac{d}{dt}\vec{y}(t) \arrow[u, bend right, "V"'] \end{tikzcd}$$

- $\frac{d}{dt}\vec{z}(t) = A\vec{z}(t)$
- $\vec{z}(t) = V\vec{y}(t)$
- $\vec{y}(t) = V^{-1}\vec{z}(t)$
- $\frac{d}{dt}\vec{z}(t) = V\frac{d}{dt}\vec{y}(t)$

Goal: $\frac{d}{dt}\vec{z}(t) = A\vec{z}(t)$

$$\frac{d}{dt}\vec{y}(t) = V^{-1}\left(A\left(V\vec{y}(t)\right)\right)$$

$$= V^{-1}AV\vec{y}(t)$$

Cheatsheet :

$$
\begin{tikzcd}
\vec{z}(t) \arrow[rr, bend left, "A"] \arrow[d, bend right, "V^{-1}"'] & & \frac{d}{dt}\vec{z}(t) \arrow[d, bend right, "V^{-1}"'] \\
\vec{y}(t) \arrow[u, bend right, "V"'] \arrow[rr] & & \frac{d}{dt}\vec{y}(t) \arrow[u, bend right, "V"']
\end{tikzcd}
$$

$\frac{d}{dt}\vec{y}(t) = V^{-1}AV\,\vec{y}(t)$

$$\frac{d}{dt}\vec{z}(t) = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} \vec{z}(t)$$ $\longrightarrow$ to solve this, use change of basis s.t.

$\vec{z}(t) = V\vec{y}(t)$ where $V = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$

discussion 2B

$\Rightarrow \vec{y}(t) = V^{-1}\vec{z}(t)$ where $V^{-1} = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}$

$$\frac{d}{dt}V\vec{y}(t) = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} V\vec{y}(t)$$

$$\frac{d}{dt}\vec{y}(t) = \underbrace{V^{-1}\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix} V}\vec{y}(t) = \overbrace{\begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix}}\vec{y}(t) \rightarrow (1)$$

e) Initial condition : $\vec{z}(0) = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$

$$\vec{y}(0) = V^{-1}\vec{z}(0)$$

$$= \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \frac{d}{dt}y_1(t) \\ \frac{d}{dt}y_2(t) \end{bmatrix} = \begin{bmatrix} -9y_1(t) \\ -2y_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y_1(0)e^{-9t} \\ y_2(0)e^{-2t} \end{bmatrix} = \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix}$$

f) $\vec{y}(t) = \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix}$ $\rightarrow$ change back to original coordinates $\vec{z}(t)$

$$\vec{z}(t) = V\vec{y}(t)$$

$$= \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -e^{-3t} \\ 3e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-3t} + 6e^{-2t} \\ -2e^{-9t} + 9e^{-2t} \end{bmatrix}$$