Discussion 2B

by Rebecca Won

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Q1. $\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix} \vec{x}(t)$

initial condition: $\vec{x}(0) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

a) $\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix} \vec{x}(t)$

Let $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

Then

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} \frac{d}{dt}x_1(t) \\ \frac{d}{dt}x_2(t) \end{bmatrix} = \begin{bmatrix} -9 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -9x_1(t) \\ -2x_2(t) \end{bmatrix}$$

$$\begin{cases} \frac{d}{dt}x_1(t) = -9x_1(t) \\ \frac{d}{dt}x_2(t) = -2x_2(t) \end{cases} \rightarrow \text{homogeneous D.E.}$$

$\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $x_1(0) = 1$ $x_2(0) = 3$

we know how to solve both!

① $\frac{d}{dt}x_1(t) = -9x_1(t) \rightarrow x_1(t) = x_1(0)e^{-9t} = e^{-9t}$

② $\frac{d}{dt}x_2(t) = -2x_2(t) \rightarrow x_2(t) = x_2(0)e^{-2t} = 3e^{-2t}$

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-9t} \\ 3e^{-2t} \end{bmatrix}$$

b) $\frac{dy_1(t)}{dt} = -5y_1(t) + 2y_2(t)$

$\frac{dy_2(t)}{dt} = 6y_1(t) - 6y_2(t)$

$\vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$

$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} \frac{d}{dt}y_1(t) \\ \frac{d}{dt}y_2(t) \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \rightarrow \frac{d}{dt}\vec{y}(t) = \overbrace{\begin{bmatrix} -5 & 2 \\ 6 & -6 \end{bmatrix}}^{A}\vec{y}(t)$

c) $\vec{\tilde{y}}(t) = \begin{bmatrix} \tilde{y}_1(t) \\ \tilde{y}_2(t) \end{bmatrix}$

$\begin{cases} y_1(t) = -\tilde{y}_1(t) + 2\tilde{y}_2(t) \\ y_2(t) = 2\tilde{y}_1(t) + 3\tilde{y}_2(t) \end{cases}$ ←

$y(t) = V\tilde{y}(t)$

$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} \tilde{y}_1(t) \\ \tilde{y}_2(t) \end{bmatrix} \Rightarrow y(t) = \overbrace{\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}}^{V}\tilde{y}(t)$

$V^{-1} = \left(\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}\right)^{-1} = \frac{1}{(-1)3-2\cdot 2}\begin{bmatrix} 3 & -2 \\ -2 & -1 \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} 3 & -2 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}$

Summary:

$$\vec{y}(t) = V\vec{\tilde{y}}(t) = \overbrace{\begin{bmatrix}-1 & 2\\ 2 & 3\end{bmatrix}}^{V}\vec{\tilde{y}}(t)$$

$$\vec{\tilde{y}}(t) = V^{-1}\vec{y}(t) = \underbrace{\begin{bmatrix}-3/7 & 2/7\\ 2/7 & 1/7\end{bmatrix}}_{V^{-1}}\vec{y}(t)$$

d) $\vec{y}(0) = \begin{bmatrix}7\\7\end{bmatrix}$

$$\vec{\tilde{y}}(0) = V^{-1}\vec{y}(0) = \begin{bmatrix}-3/7 & 2/7\\ 2/7 & 1/7\end{bmatrix}\begin{bmatrix}7\\7\end{bmatrix} = \begin{bmatrix}-3+2\\ 2+1\end{bmatrix} = \begin{bmatrix}-1\\3\end{bmatrix}$$

e) $\frac{d}{dt}\vec{y}(t) = \begin{bmatrix}-5 & 2\\ 6 & -6\end{bmatrix}\vec{y}(t)$ ← in terms of $\vec{\tilde{y}}(t)$

$\vec{y}(t) = V\cdot\vec{\tilde{y}}(t)$

$$\Rightarrow \frac{d}{dt}V\vec{\tilde{y}}(t) = \begin{bmatrix}-5 & 2\\ 6 & -6\end{bmatrix}V\cdot\vec{\tilde{y}}(t)$$

$$V\frac{d}{dt}\vec{\tilde{y}}(t) = \begin{bmatrix}-5 & 2\\ 6 & -6\end{bmatrix}V\vec{\tilde{y}}(t)$$

$$\frac{d}{dt}\vec{\tilde{y}}(t) = V^{-1}\begin{bmatrix}-5 & 2\\ 6 & -6\end{bmatrix}V\vec{\tilde{y}}(t)$$

$$V^{-1}\begin{bmatrix}-5 & 2\\ 6 & -6\end{bmatrix}V = \begin{bmatrix}-\frac{3}{7} & \frac{2}{7}\\ \frac{2}{7} & \frac{1}{7}\end{bmatrix}\begin{bmatrix}-5 & 2\\ 6 & -6\end{bmatrix}\begin{bmatrix}-1 & 2\\ 2 & 3\end{bmatrix} = \begin{bmatrix}-9 & 0\\ 0 & -2\end{bmatrix}$$

$\Rightarrow \frac{d}{dt}\vec{\tilde{y}}(t) = \begin{bmatrix}-9 & 0\\ 0 & -2\end{bmatrix}\vec{\tilde{y}}(t)$ → homogeneous D.E. same as part (a)

f) $\frac{d}{dt}\vec{\tilde{y}}(t) = \begin{bmatrix} -9 & 0 \\ 0 & 2 \end{bmatrix} \vec{\tilde{y}}(t)$

$\vec{\tilde{y}}(t) = \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix}$ from part (a)

$\vec{y}(t) = V\vec{\tilde{y}}(t)$

$= \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -e^{-9t} \\ 3e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-9t} + 6e^{-2t} \\ -2e^{-9t} + 9e^{-2t} \end{bmatrix}$