Discussion 2A

Notes made by

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EECS 16B Su22

Q1.

Goal: Discretization

Discretization by sampling

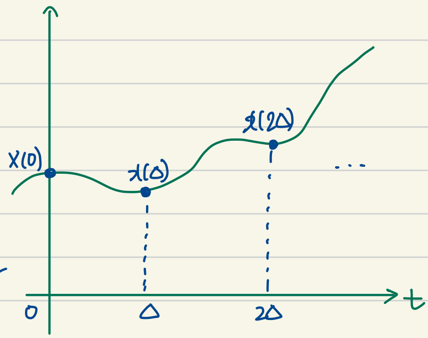
$\boxed{\text{Analog}} \longrightarrow$ Digital

ex) sensor signal
→ continuous time

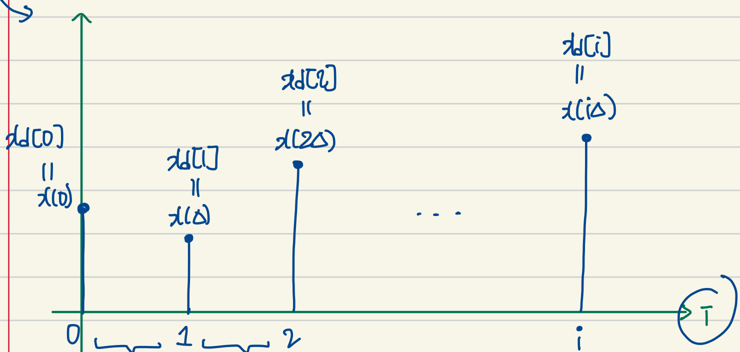
computer logic
→ discrete time

⟨ Discretization by Sampling ⟩

– continuous time



– discrete time



$t \in [i\Delta, (i+1)\Delta) \longrightarrow x_d[i] = x(i\Delta)$

i.e. between $T=0$ & $T=1$, we use the value at $T=0$ i.e. $x_d[0]$

$\Rightarrow$ our input is also discrete

i.e. $u(t) = u(i\Delta) = u_d[i]$ for $t \in [i\Delta, (i+1)\Delta)$

a) Recall:

$$\frac{d}{dt}x(t) = \lambda x(t) + bu(t)$$

$$x(t) = e^{\lambda(t-t_0)}x(t_0) + b\int_{t_0}^{t} \overset{u_d[i]}{u(\theta)}\, e^{\lambda(t-\theta)}\,d\theta \quad , \quad \overset{i\Delta}{t_0} = \text{starting time}$$

$$x(t) = e^{\lambda(t-t_0)}x(t_0) + b\int_{t_0}^{t} u_d[i]\, e^{\lambda(t-\theta)}\,d\theta$$

$$\rightarrow x(t) = e^{\lambda(t-i\Delta)}x(t_0) + b\,u_d[i]\int_{i\Delta}^{t} e^{\lambda(t-\theta)}\,d\theta$$

Goal: $x_d[i+1] = x((i+1)\Delta) = ??\, x_d[i]$

$$\star\; x((i+1)\Delta) = e^{\lambda[(i+1)\Delta - i\Delta]}x_d[i] + b\,u_d[i]\int_{i\Delta}^{(i+1)\Delta} e^{\lambda((i+1)\Delta-\theta)}\,d\theta$$

$$= e^{\lambda\Delta}x_d[i] + b\,u_d[i]\left(\frac{e^{\lambda\Delta}-1}{\lambda}\right)$$

Result: $\frac{d}{dt}x(t) = \lambda x(t) + b\,u_d[i]$ $\quad t \in [i\Delta, (i+1)\Delta)$, $x(t)$ continuous

$$x_d[i+1] = x((i+1)\Delta) = e^{\lambda\Delta}x_d[i] + b\cdot u_d[i]\left(\frac{e^{\lambda\Delta}-1}{\lambda}\right)$$

b) From part a): $x_d[i+1] = \overset{\beta}{\overset{\alpha}{\overset{\|}{e^{\lambda\Delta}}}} x_d[i] + b u_d[i]\left(\overset{\beta}{\overset{\|}{\frac{e^{\lambda\Delta}-1}{\lambda}}}\right)$ ←

$$\Rrightarrow \quad \alpha = e^{\lambda\Delta}$$
$$\beta = \frac{e^{\lambda\Delta}-1}{\lambda}$$

Tip: Try unrolling the equation and find the pattern

c) $x_d[i+1] = \alpha x_d[i] + \beta u_d[i]$ → recurring equations

$$x_d[1] = \alpha x_d[0] + \beta u_d[0]$$
$$x_d[2] = \alpha x_d[1] + \beta u_d[1]$$
$$= \alpha(\alpha x_d[0] + \beta u_d[0]) + \beta u_d[1]$$
$$= \alpha^2 x_d[0] + \beta(\alpha u_d[0] + u_d[1])$$

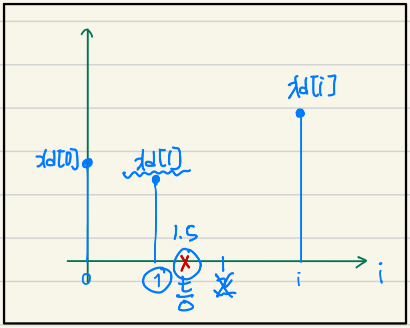
$$x_d[3] = \alpha x_d[2] + \beta u_d[2]$$
$$= \alpha(\alpha^2 x_d[0] + \beta(\alpha u_d[0] + u_d[1])) + \beta u_d[2]$$
$$= \alpha^3 x_d[0] + \beta(\alpha^2 u_d[0] + \alpha u_d[1] + u_d[2])$$

Pattern: $x_d[i] = \alpha^i x_d[0] + \beta \sum_{k=0}^{i-1} \alpha^{i-1-k} u_d[k]$

d) We use i iff $t \in [i\Delta, (i+1)\Delta)$

$$\underset{1.5}{\underset{\|}{\frac{t}{\Delta}}} \in [\underset{1}{\underset{\|}{i}}, \underset{\cancel{2}}{\underset{\|}{i+1}}) \rightarrow i = \lfloor 1.5 \rfloor = 1$$

$$i = \left\lfloor \frac{t}{\Delta} \right\rfloor$$



$$i = \left\lfloor \frac{t}{\Delta} \right\rfloor$$

e) $-$ 1b: $\alpha = e^{\lambda\Delta}$, $\beta = \dfrac{b(e^{\lambda\Delta}-1)}{\lambda}$ (1)

$-$ 1c: $x_d[i] = \alpha^i x_d[0] + \beta \sum_{k=0}^{i-1} \alpha^{i-1-k} u[k]$ (2)

$-$ 1d: $i = \left\lfloor \frac{t}{\Delta} \right\rfloor$ (3)

$x(t) \approx x_d[i]$

$$= (e^{\lambda\Delta})^i x_d[0] + \frac{b(e^{\lambda\Delta}-1)}{\lambda} \sum_{k=0}^{i-1} (e^{\lambda\Delta})^{i-1-k} u[k]$$

$$= (e^{\lambda\Delta})^{\lfloor \frac{t}{\Delta} \rfloor} x_d[0] + \frac{b(e^{\lambda\Delta}-1)}{\lambda} \sum_{k=0}^{\lfloor \frac{t}{\Delta} \rfloor - 1} (e^{\lambda\Delta})^{\lfloor \frac{t}{\Delta} \rfloor - 1 - k} u[k]$$