EECS 16B Designing Information Devices and Systems II
Summer 2020 UC Berkeley

Controls Review

1. **Airplane Discretization**

   In this question we will explore briefly a simplified linear model of an airplane control model. First let's define the variables we will be working with, in reference with figure 1:

   (i) $\alpha =$ Angle of Attack (angle the plane makes with the direction of wind)
   (ii) $\theta =$ Pitch Angle (angle of the plane with respect to the horizontal)
   (iii) $u =$ Elevator angle (used to control the aircraft's pitch)

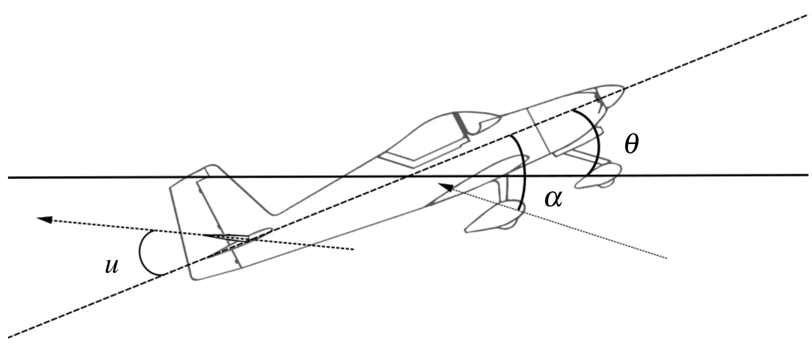


Figure 1: Simplified Airplane model

Now, consider the following (simplified) continuous time system:

$$\frac{d}{dt}\alpha = 5\alpha - \frac{d}{dt}\theta + c_1\delta \quad (1)$$

$$\frac{d^2}{dt^2}\theta = -\alpha + \frac{d}{dt}\theta + c_2\delta \quad (2)$$

(a) For the system of differential equations given, **write the matrix differential equation as**

$$\frac{d}{dt}\vec{x} = A\vec{x} + B\delta$$

Solution: We can define our state vector as $\vec{x} = \begin{bmatrix} \alpha \\ \theta \\ \frac{d\theta}{dt} \end{bmatrix}$ to get the equations

$$\frac{d}{dt}x_1 = 5x_1 - x_3 + c_1\delta$$
$$\frac{d}{dt}x_2 = x_3$$
$$\frac{d}{dt}x_3 = -x_1 + x_3 + c_2\delta$$

This gives us the following state-matrices

$$A = \begin{bmatrix} 5 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} c_1 \\ 0 \\ c_2 \end{bmatrix} \tag{3}$$

Alternatively since the state θ is never used, we can pick $\vec{x} = \begin{bmatrix} \alpha \\ \frac{d\theta}{dt} \end{bmatrix}$ with the following state-matrices.

$$A = \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{4}$$

(b) Now, assume for some specific component values we get the following differential equation:

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\vec{x}(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix}u(t). \tag{5}$$

Unfortunately, we are unable to measure our state vector continuously. Suppose that we sample the system with some sampling interval T. Let us discretize the above system. Assume that we use piecewise constant voltage inputs $u(t) = u[k]$ for $t \in [kT, (k+1)T)$.

Calculate a discrete-time system for Equation (5)'s continuous-time vector system in the form:

$$\vec{x}[k+1] = A_d\vec{x}[k] + \vec{b}_d[k].$$

Solution:

One way to discretize this system is to change coordinates to the eigenbasis, and discretize the individual scalar equations. Having done so, we can change coordinates back to the standard basis.

Since A is diagonalizable, we can write $A = V\Lambda V^{-1}$, substitute into our differential equation, to get:

$$\frac{d}{dt}\vec{x} = V\Lambda V^{-1}\vec{x} + \vec{b}u(t)$$
$$\frac{d}{dt}V^{-1}\vec{x} = \Lambda V^{-1}\vec{x} + V^{-1}\vec{b}u(t)$$
$$\frac{d}{dt}\vec{z} = \Lambda\vec{z} + V^{-1}\vec{b}u(t)$$

Writing, $\vec{z} = V^{-1}\vec{x}$, we can diagonlize the system. We can also compute the following matrices

$$V = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \qquad V^{-1}\vec{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

This gives us two differential equations

$$\frac{d}{dt}z_1(t) = -2z_1(t) - 2u(t)$$
$$\frac{d}{dt}z_2(t) = -z_2(t) - 2u(t)$$

Recall that the following scalar differential equation can be discretized as the following

$$\frac{d}{dt}x(t) = \lambda x(t) + bu(t) \implies x[k+1] = e^{\lambda T}x[k] + b\frac{e^{\lambda T} - 1}{\lambda}u[k]$$

Hence, we can discretize the system in this diagonal space, giving us

$$z_1[k+1] = e^{-2T}z_1[k] + (e^{-2T} - 1)u[k]$$
$$z_2[k+1] = e^{-T}z_2[k] + 2(e^{-T} - 1)u[k]$$

This gives us the equation

$$\vec{z}[k+1] = \begin{bmatrix} e^{-2T} & 0 \\ 0 & e^{-T} \end{bmatrix}\vec{z}[k] + \begin{bmatrix} e^{-2T} - 1 \\ 2e^{-T} - 2 \end{bmatrix}u[k].$$

Changing coordinates back to $\vec{x}$ we see that

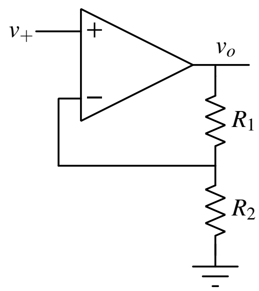
$$A_d = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} e^{-2T} & 0 \\ 0 & e^{-T} \end{bmatrix}\begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$$
$$\vec{b}_d = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} e^{-2T} - 1 \\ 2e^{-T} - 2 \end{bmatrix}$$

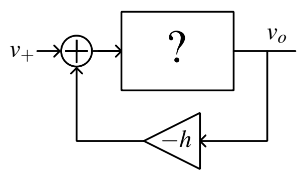
Multiplying out the matrices, we get:

$$A_d = \begin{bmatrix} 2e^{-T} - e^{-2T} & e^{-T} - e^{-2T} \\ 2e^{-2T} - 2e^{-T} & 2e^{-2T} - e^{-T} \end{bmatrix}$$
$$\vec{b}_d = \begin{bmatrix} e^{-2T} - 2e^{-T} + 1 \\ 2e^{-T} - 2e^{-2T} \end{bmatrix}$$

2. Feedback Control of Op-Amps (X pts)

You have seen op-amps in negative feedback many times, and you have learned about feedback control. You may not have realized it yet, but these are actually related to each other.





Here, we introduce a dynamic model for a non-ideal op-amp in negative feedback:

$$\frac{d}{dt}v_o(t) = -v_o(t) + Gu(t)$$

where v_o is the output voltage, and $u(t) = \left(v_+(t) - v_-(t)\right)$ where v_+ and v_- are the voltages at the positive and negative inputs respectively, and $G > 2$ is a parameter that defines the op-amp's behavior.

(a) (X pts) Given the dynamic model for the nonideal op-amp, **assuming v_+ and v_- are not changing, for what value of v_o will v_o not be changing?** Your answer should depend on v_+, v_-, G.

Solution: To find a steady-state value, set $\frac{d}{dt}v_o(t) = 0$, so the whole dynamic model simply becomes $v_o(t) = Gu(t) = G\left(v_+(t) - v_-(t)\right)$.

(b) (X pts) In the above, $v_-(t) = hv_o(t)$. **Pick values for the resistors R_1 and R_2 so that h equals $\frac{1}{2}$.**

Solution: The h from a resistive divider will be $\frac{R_2}{R_1+R_2}$, so to set that equal to $\frac{1}{2}$, $2R_2 = R_1 + R_2$, so the resistors must be equal.

It doesn't much matter which values you pick provided $R_2 = R_1$, but practical considerations tend to set the value somewhere between 10k and 10M.

(c) (8 pts) Suppose we place the nonideal op-amp in resistive negative feedback using a voltage divider whose ratio is $h = \frac{1}{2}$, in other words set

$$u(t) = v_+(t) - hv_o(t).$$

Write out the new differential equation that relates $v_o(t)$ to $v_+(t)$. Is this system stable? Briefly state why or why not.

Solution: Plugging the given value for u into the dynamical model,

$$\frac{d}{dt}v_o(t) = -v_o(t) + G\left(v_+(t) - hv_o(t)\right) = -\left(1 + Gh\right)v_o(t) + Gv_+(t)$$

The eigenvalue has therefore changed from -1 to $-1 - Gh$, which means the system will be stable provided $G > -2$. Since we have been given that $G > 2$, the eigenvalue is definitely negative and there is no stability issue.

(d) (4 pts) If we had swapped the roles of the positive and negative terminals of the op-amp (i.e. had hooked the nonideal op-amp up in positive feedback so that $u(t) = hv_o(t) - v_-(t)$.) **would the resulting closed-loop system have been stable? Briefly state why or why not.**

Solution: Plugging in the new given value for u into the dynamic model, this time we get

$$\frac{d}{dt}v_o(t) = -v_o(t) + G\left(hv_o(t) - v_-(t)\right) = (Gh-1)\,v_o(t) - v_-(t)$$

This eigenvalue is $Gh-1$, which (because we know $G > 2$) is positive, so the system is not stable.

(e) (10 pts) For the closed-loop system in negative feedback (from part (c)) using resistor values that set $h = \frac{1}{2}$, assume that the output voltage starts at 0V at time 0 and that $v_+(t)$ was 0V but then jumps up to 1V at time 0. **How long will it take for $v_o(t)$ to reach 1V?** Your answer should be in terms of G.

Solution: Since our dynamical model is a simple first-order differential equation of the style we have seen repeatedly in 16, we can explicitly solve it without too much trouble.

The first step is to figure out what the formal steady-state would be after the transition — i.e. at what v_o would the derivative be zero. By inspection, the answer is $\frac{Gv_+}{Gh+1}$.

Now, we can change variables to see the transient behavior and consider $\tilde{v}_o(t) = v_o(t) - \frac{Gv_+}{Gh+1}$. It is clear that $\frac{d}{dt}\tilde{v}_o(t) = \frac{d}{dt}v_o(t)$ and hence the relevant differential equation is:

$$\frac{d}{dt}\tilde{v}_o(t) = -(1+Gh)\,v_o(t) - v_+ = -(1+Gh)\,\tilde{v}_o(t).$$

This has the solution $\tilde{v}_o(t) = e^{-(1+Gh)t}\tilde{v}_o(0)$ where the initial condition for the transient is clearly $\tilde{v}_o(0) = -\frac{Gv_+}{1+Gh}$. Putting it all together, we get:

$$v_o(t) = \frac{G}{1+Gh}\left(1 - \mathrm{e}^{-(1+Gh)t}\right)v_+$$

Then substituting in the parameters $v_+ = 1$, $h = \frac{1}{2}$, we can set $v_o(t) = 1$ and solve for t:

$$\begin{aligned}
1 &= \frac{2G}{2+G}\left(1 - \mathrm{e}^{-(1+G/2)t}\right) \\
\frac{2+G}{2G} &= 1 - \mathrm{e}^{-(1+G/2)t} \\
\mathrm{e}^{-(1+G/2)t} &= 1 - \frac{2+G}{2G} = \frac{G-2}{2G} \\
\left(1+\frac{G}{2}\right)t &= \ln\frac{2G}{G-2} \\
t &= \frac{\ln\frac{2G}{G-2}}{1+G/2}
\end{aligned}$$

It is pretty common in practice (because G is usually very large, typically something like a million) to assume the final value is exactly 2; under that assumption we can solve the problem much more easily by observing that the t we're looking for is halfway between our initial and final values: $\mathrm{e}^{\lambda t} = \frac{1}{2}$, so

$$t = \frac{-\ln 2}{\lambda} = \frac{\ln 2}{1+G/2}$$

That's pretty close to the value from the real method, and becomes closer as G gets larger.

(f) (4 pts) **What happens to the answer of the previous part if $G \to \infty$?**

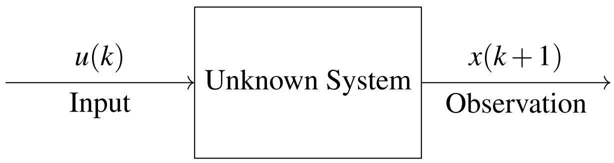
Solution: Hopefully it is intuitively clear that it should turn to 0; the more negative our eigenvalue, the faster the system responds, so when our eigenvalue goes to $-\infty$, the system should respond instantly.

If we actually mathematically take the limit of the answer in the previous part, we do in fact get 0 since the log in the numerator is approaching a limit while the G in the denominator is growing. So both intuition and math agree.

3. System Identification

In this question, we will take a look at how to **identify** a system by taking experimental data taken from a (presumably) linear system to learn a discrete-time linear model for it using the least-squares.

The visual below shows our experimental procedure of giving an input sequence $u(k)$ and taking observations $x(k+1)$.



We will start by assuming the model for the system is $x(k+1) = \alpha x(k) + \beta u(k) + e(k)$.

(a) Given the sequence of inputs $u(0), u(1), u(2)$ and initial state $x(0)$ set up a Least Squares problem to estimate α and β.

Solution: We will feed in the sequence of inputs $u(0), u(1), u(2)$ and observe states $x(1), x(2), x(3)$. Given the observations, we can set up the following system of equations:

$$x(1) = \alpha x(0) + \beta u(0) + e(0)$$
$$x(2) = \alpha x(1) + \beta u(1) + e(1)$$
$$x(3) = \alpha x(2) + \beta u(2) + e(2)$$

This can then be written as the following matrix-vector equation:

$$\begin{bmatrix} x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} x(0) & u(0) \\ x(1) & u(1) \\ x(2) & u(2) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} e(0) \\ e(1) \\ e(2) \end{bmatrix}$$

Which can equivalently be written as:

$$\vec{y} = D\vec{p} + \vec{e}$$

The Least Squares problem minimizes the norm of the vector $\vec{e}$ and can be written as

$$\min_{\vec{p}} \|\vec{e}\| = \min_{\vec{p}} \|\vec{y} - D\vec{p}\|$$

(b) When will the Least Squares problem set up in part (a) have a unique solution?

Solution: In order to use Least Squares, the matrix $D^T D$ must be invertible. This is equivalent to saying that the matrix D is full rank or has linearly independent columns.

(c) Using this fact from the previous part, what is the minimum number of measurements we need to make in order to set up a Least Squares problem?

Solution: If we make n measurements, the matrix D will be of size $n \times 2$. The matrix D must have at least 2 rows for it to be full rank. Therefore, we must take at least 2 measurements ot set up a Least Squares problem. Note that this does not mean any 2 measurements will give a unique least square solution.

(d) Given the initial state $x(0) = 1$, input sequence $u(0) = 1, u(1) = 1$ and observations $x(1) = -1, x(2) = 3$, provide estimates for α and β.

Solution: We first create our D matrix and $\vec{y}$ vector.

$$D = \begin{bmatrix} x(0) & u(0) \\ x(1) & u(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} x(1) \\ x(2) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Then the least-squares estimate $\hat{\vec{p}}$ can be computed as follows:

$$\hat{\vec{p}} = (D^T D)^{-1} D^T \vec{y} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

(e) Now let's consider the model $x(k+1) = \alpha x(k) + \beta_0 + \beta_1 u(k) + e(k)$. Give an input sequence of length 3 where Least Squares will fail.

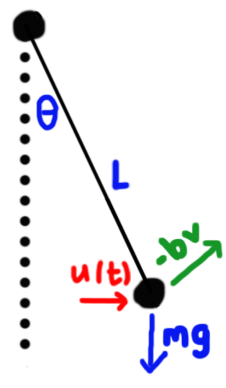
Solution: The data matrix D will be:

$$D = \begin{bmatrix} x(0) & 1 & u(0) \\ x(1) & 1 & u(1) \\ x(2) & 1 & u(2) \end{bmatrix}$$

Least Squares will fail when the matrix D is not full rank. An input sequence that will make Least Squares fail is $u(0) = 1, u(1) = 1, u(2) = 1$. Another example is $u(0) = x(0), u(1) = x(1), u(2) = x(2)$.

4. Inverted Pendulum

We will now take a look at the same pendulum from the previous question but around a new equilibrium point $\vec{x}^* = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$ and $u^* = 0$.



To recall, the kinetics of this pendulum over time can be represented by the following differential equation:

$$\frac{d^2\theta}{dt^2} + \frac{b}{m}\frac{d\theta}{dt} + \frac{g}{L}\sin\theta = \frac{\cos\theta}{mL}u \tag{6}$$

(a) **What is the linearized system around this operating point?**

Solution: We can use the Jacobian matrix from before

$$J_{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos x_1 - \frac{\sin x_1}{mL}u & -\frac{b}{m} \end{bmatrix} \text{ and } J_u = \begin{bmatrix} 0 \\ \frac{\cos x_1}{mL} \end{bmatrix}$$

Evaluating at our new equilibrium points $\vec{x}^*, u^*$ we get

$$J_{\vec{x}}|_{\vec{x}^*} = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & -\frac{b}{m} \end{bmatrix} \text{ and } J_u|_{u^*} = \begin{bmatrix} 0 \\ -\frac{1}{mL} \end{bmatrix}$$

This means our new linearized system is

$$\frac{d}{dt}\vec{x}_\ell = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & -\frac{b}{m} \end{bmatrix}\vec{x}_\ell + \begin{bmatrix} 0 \\ -\frac{1}{mL} \end{bmatrix}u$$

Where $\vec{x}_\ell = \vec{x} - \vec{x}^*$.

(b) **Is this linearized system stable when $b > 0$?**

Solution: We can compute the characteristic polynomial to be $\lambda^2 + \frac{b}{m}\lambda - \frac{g}{L}$. This means that the eigenvalues will be:

$$\lambda = -\frac{b}{2m} \pm \frac{1}{2}\sqrt{\left(\frac{b}{m}\right)^2 + \frac{4g}{L}}$$

Since m, g, L are all positive and $\sqrt{\left(\frac{b}{2m}\right)^2 + \frac{g}{L}} > \frac{b}{2m}$, there will be one eigenvalue with real part greater than zero meaning the system is unstable.

(c) **Show that the system can be put in CCF for the given physical values:** $b = 2,\ m = 2,\ g = 10,\ L = \frac{1}{2}$. *Hint: You will have to make some modification to the input.*

Solution: If we reverse the direction of our input letting $u_{new} = -u$ then we see that the system is indeed in Controllable Canonical Form:

$$\frac{d}{dt}\vec{x}_\ell(t) = \begin{bmatrix} 0 & 1 \\ 20 & -1 \end{bmatrix} \vec{x}_\ell(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{new}(t)$$

(d) **Is this linearized system controllable?**

Solution: We can claim that this system is controllable, since it is in Controllable Canonical Form when $mL = 1$. Alternatively we can also compute the controllability matrix as:

$$\mathscr{C} = \begin{bmatrix} 0 & -\frac{1}{mL} \\ -\frac{1}{mL} & \frac{b}{m^2 L} \end{bmatrix}$$

Since $\frac{1}{mL}$ cannot be zero, the second column cannot be a scalar multiple of the first. Therefore, the controllability matrix is full rank, and the system is controllable.

(e) **Design a feedback controller so that the eigenvalues of the system will be** $\lambda_1 = -2,\ \lambda_2 = -3$.

Solution: Setting $u_{new} = K\vec{x}(t)$ as the input, where $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, the closed-loop matrix will be:

$$A_{cl} = \begin{bmatrix} 0 & 1 \\ 20 + k_1 & -1 + k_2 \end{bmatrix}$$

We know that the characteristic polynomial of this matrix will be $\lambda^2 - (-1 + k_2)\lambda - (20 + k_1)$. To make the eigenvalues -2 and -3, we want our characteristic polynomial to be $\lambda^2 + 5\lambda + 6$ so picking $k_1 = -26$ and $k_2 = -4$ will suffice. Note that this is equivalent to picking $u = -K\vec{x} = \begin{bmatrix} 26 & 4 \end{bmatrix} \vec{x}$

(f) **Suppose that we our feedback controller is limited and can only give feedback to the θ variable. In other words, $K = \begin{bmatrix} k & 0 \end{bmatrix}$. Is it still possible to pick eigenvalues that will make the system stable?**

Solution: This is equivalent to the scenario above in which $k_1 = k$ and $k_2 = 0$ meaning the characteristic polynomial will be

$$\lambda^2 + \lambda - (20 + k_1)$$

Therefore it is possible to stabilize the system is we pick $k_1 < -20$. An example will be $k_1 = -\frac{81}{4}$ and there would be a repeated eigenvalue of $\lambda = -\frac{1}{2}$.

5. Controllable Car Form

Recall the car-system can represent the system with the following differential equation with position $p(t)$ and velocity $v(t)$

$$\frac{d}{dt}\begin{bmatrix}p(t)\\ v(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}p(t)\\ v(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{M}\end{bmatrix}u(t) \tag{7}$$

Assuming $M = 1$ let us discretize this system with sampling rate $T = 1$.

$$\begin{bmatrix}p[n+1]\\ v[n+1]\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}p[n]\\ v[n]\end{bmatrix} + \begin{bmatrix}\frac{1}{2}\\ 1\end{bmatrix}u[n] \tag{8}$$

Since the discretized system is unstable, we will try to build a feedback controller $u[n] = -K\vec{x}[n]$ to stabilize its eigenvalues.

(a) **What is the closed loop matrix A_{cl}?**

Solution: The closed loop matrix will be $A_{cl} = A - BK$.

(b) **Verify that this system is controllable.**

Solution: The system is controllable since

$$C = \begin{bmatrix}B & AB\end{bmatrix} = \begin{bmatrix}0.5 & 1.5\\ 1 & 1\end{bmatrix} \tag{9}$$

(c) **Find the transformation T that puts this system in Controllable Canonical Form**

Solution: The characteristic polynomial is $\lambda^2 - 2\lambda + 1$ so the A and B matrices in CCF are

$$A_c = \begin{bmatrix}0 & 1\\ -1 & 2\end{bmatrix} \qquad B_c = \begin{bmatrix}0\\ 1\end{bmatrix} \tag{10}$$

The transformation T to CCF can be computed as

$$T = \tilde{C}C^{-1} = \begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}0.5 & 1.5\\ 1 & 1\end{bmatrix}^{-1} = \begin{bmatrix}1 & -0.5\\ 1 & 0.5\end{bmatrix}$$

(d) **Pick values for K_c so that places both eigenvalues at $\lambda = \frac{1}{2}$ in the controller basis.** **Solution:** The desired characteristic polynomial that puts both eigenvalues at $\lambda = \frac{1}{2}$ is

$$\lambda^2 - \lambda + \frac{1}{4}$$

The characteristic polynomial of A_{cl} is

$$\lambda^2 + (-2 + k_1)\lambda + (1 + k_0)$$

Therefore, picking $K_c = \begin{bmatrix}-0.75 & 1\end{bmatrix}$ in the controller basis will set the eigenvalues at $\frac{1}{2}$.

(e) **Find the controller $u[n] = -K\vec{x}[n]$ in the standard basis.**

Solution: To convert back to standard coordinates, $u[n] = -K_c\vec{z}[n] = -K_cT\vec{x}[n]$ which tells us that $K = K_cT$.

$$K = K_cT = \begin{bmatrix}-0.75 & 1\end{bmatrix}\begin{bmatrix}1 & -0.5\\ 1 & 0.5\end{bmatrix} = \begin{bmatrix}\frac{1}{4} & \frac{7}{8}\end{bmatrix}$$

6. Day at the Races

Forrest is building his SIXT33N Car to compete at the annual 16B racing competition. In this competition, cars can take any path, but they must stop exactly at the finish line.

Luckily Forrest has found a shortcut in the track so that the car can move in a straight line from start to finish. Therefore, he will use the following discrete-time model giving inputs to his car that weighs 10 kg every 1 s.

$$\begin{bmatrix} p[n+1] \\ v[n+1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p[n] \\ v[n] \end{bmatrix} + \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix} u[n] \tag{11}$$

(a) Assuming the car starts at rest, the finish line is 40 m away. **What inputs can Forrest give to reach the finish line in** 2 s?

Solution: In two time steps, the dynamics of our system tell us that

$$\vec{x}[2] = A^2\vec{x}[0] + \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u[1] \\ u[0] \end{bmatrix}$$

Our target state is $\vec{x}[2] = \begin{bmatrix} 40 & 0 \end{bmatrix}^T$ meaning the inputs will be

$$\begin{bmatrix} u[1] \\ u[0] \end{bmatrix} = C^{-1} \begin{bmatrix} 40 \\ 0 \end{bmatrix}$$

Plugging the values in, we get $u[0] = 400, u[1] = -400$.

(b) Upon giving these inputs, Forrest's car explodes due to the inputs being too strong. Therefore, he builds a new car and now tries to reach the finish line in 5 s.
Set up an optimization problem of the form

$$\min_{\vec{w} \in \mathbb{R}^5} \|\vec{w}\|^2 \ \text{ subject to } H\vec{w} = \vec{y} \tag{12}$$

so that the car can reach the finish line with minimum energy.

Solution: Let's start by writing out $\vec{x}[5]$ in terms of the previous states

$$\begin{aligned} \vec{x}[5] &= A^5\vec{x}[0] + A^4\vec{b}u[0] + A^3\vec{b}u[1] + A^2\vec{b}u[2] + A\vec{b}u[3] + \vec{b}u[4] \\ &= \begin{bmatrix} A^4\vec{b} & A^3\vec{b} & A^2\vec{b} & A\vec{b} & \vec{b} \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \\ u[4] \end{bmatrix} \\ \vec{y} &= H\vec{w} \end{aligned}$$

Since $H = \begin{bmatrix} A^4\vec{b} & A^3\vec{b} & A^2\vec{b} & A\vec{b} & \vec{b} \end{bmatrix}$ is a wide matrix of full-rank, we see that the system of equations $H\vec{w} = \vec{y}$ has infinite solutions. Our goal is to reach $\vec{y} = \begin{bmatrix} 40 \\ 0 \end{bmatrix}$ with minimum energy or in otherwords, we would like to minimize $\|\vec{w}\|^2 = w_1^2 + \ldots + w_5^2$ over all solutions of $H\vec{w} = \vec{y}$.
This is equivalent to an optimization problem of the form

$$\min_{\vec{w} \in \mathbb{R}^5} \|\vec{w}\|^2 \ \text{ subject to } H\vec{w} = \vec{y}$$

(c) Forrest creates 5 checkpoints $\vec{r}[1],\ldots,\vec{r}[5]$ to ensure that his car is moving in the right direction. **Set up an optimization problem of the form**

$$\min_{\vec{w}\in\mathbb{R}^{12}} \|\vec{w}\|^2 \text{ subject to } G\vec{w}=\vec{y} \tag{13}$$

that minimizes the both the energy and the squared distances from each checkpoint.
Hint: Define distances $\vec{d}[k]=\vec{x}[k]-\vec{r}[k]$ and write out the State-Space equations for $\vec{x}[k]$. Then try to create a vector $\vec{w}\in\mathbb{R}^{12}$ that represents both the distances and inputs.

Solution: Let's represent $\vec{d}$ as the distance between $\vec{x}$ and $\vec{r}$ at each time-step $n=1$ to 5. This gives us 5 equations of the form

$$\begin{aligned}\vec{d}[1]&=\vec{x}[1]-\vec{r}[1]\\&\vdots\\\vec{d}[5]&=\vec{x}[5]-\vec{r}[5]\end{aligned}$$

For each $\vec{x}[n]$, we can write them out in terms of the inputs $u[0],\ldots,u[n-1]$.

$$\begin{aligned}\vec{x}[1]&=A\vec{x}[0]+Bu[0]\\\vec{x}[2]&=A^2\vec{x}[0]+ABu[0]+Bu[1]\\&\vdots\\\vec{x}[5]&=A^5\vec{x}[0]+A^4Bu[0]+\cdots+Bu[4]\end{aligned}$$

This can all be massaged into the following constraints

$$\begin{aligned}\vec{r}[1]&=Bu[0]-\vec{d}[1]\\&\vdots\\\vec{r}[5]&=Bu[4]+\ldots+A^4Bu[0]-\vec{d}[5]\end{aligned}$$

Since our goal is to minimize the following sum,

$$\sum_{k=1}^{5}\left\|\vec{d}[k]\right\|^2+\sum_{n=0}^{4}u[n]^2$$

we can create a vector $\vec{w}\in\mathbb{R}^{12}$ of the form below and write out our constraints as

$$\underbrace{\begin{bmatrix}B & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ AB & 0 & B & -I & 0 & 0 & 0 & 0 & 0 & 0\\ A^2B & 0 & AB & 0 & B & -I & 0 & 0 & 0 & 0\\ A^3B & 0 & A^2B & 0 & AB & 0 & B & -I & 0 & 0\\ A^4B & 0 & A^3B & 0 & A^2B & 0 & AB & 0 & B & -I\end{bmatrix}}_{G}\underbrace{\begin{bmatrix}u[0]\\\vec{d}[1]\\u[1]\\\vec{d}[2]\\u[2]\\\vec{d}[3]\\u[3]\\\vec{d}[4]\\u[4]\\\vec{d}[5]\end{bmatrix}}_{\vec{w}}=\underbrace{\begin{bmatrix}\vec{r}[1]\\\vec{r}[2]\\\vec{r}[3]\\\vec{r}[4]\\\vec{r}[5]\end{bmatrix}}_{\vec{y}}$$

(d) Forrest notices that Simon's car is sending out distrubances that prevents his car from moving straight. Therefore, Forrest sets up a new state-space model with $\vec{d}[n] = \vec{x}[n] - \vec{r}[n]$.

$$\vec{d}[n+1] = A\vec{d}[n] + B(u[n] - u^*[n])$$

where $u^*[n]$ are the control inputs from the previous part that reach the target with minimum energy. **Explain how Forrest can design a feedback controller to ensure that the car will be close to the checkpoints** $\vec{r}[n]$.

Solution: By defining the input to the system as $w[n] = u[n] - u^*[n]$, the state-space model for $\vec{d}$ can be written as

$$\vec{d}[n+1] = A\vec{d}[n] + Bw[n]$$

In order to make the car close to the checkpoints $\vec{r}[n]$, we need to ensure that $\vec{d}[n] = \vec{x}[n] - \vec{r}[n]$ is close to zero. This is equivalent to saying that we want this system to be stable.

Therefore, if Forrest designs a feedback controller $w[n] = -K\vec{d}[n]$ that sets all of the eigenvalues of $A - BK$ to have magnitude less than 1, $\vec{d}[n]$ will converge to 0 meaning the car is near the checkpoints.

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