EECS 16B Designing Information Devices and Systems II
Summer 2020 UC Berkeley

Controls Review

1. Airplane Discretization

In this question we will explore briefly a simplified linear model of an airplane control model. First let's define the variables we will be working with, in reference with figure 1:

(i) $\alpha =$ Angle of Attack (angle the plane makes with the direction of wind)

(ii) $\theta =$ Pitch Angle (angle of the plane with respect to the horizontal)

(iii) $u =$ Elevator angle (used to control the aircraft's pitch)

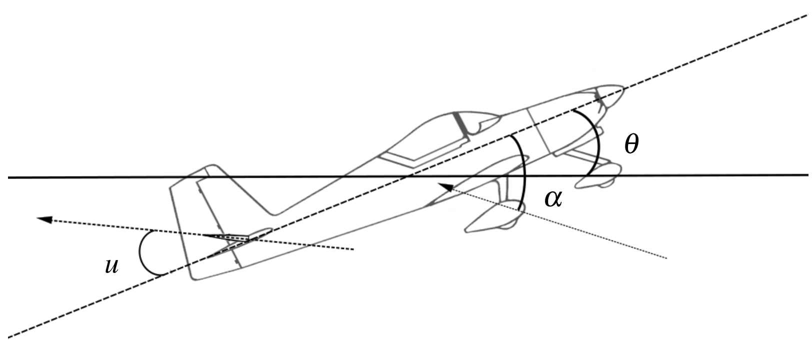


Figure 1: Simplified Airplane model

Now, consider the following (simplified) continuous time system:

$$\frac{d}{dt}\alpha = 5\alpha - \frac{d}{dt}\theta + c_1\delta \tag{1}$$

$$\frac{d^2}{dt^2}\theta = -\alpha + \frac{d}{dt}\theta + c_2\delta \tag{2}$$

(a) For the system of differential equations given, **write the matrix differential equation as**

$$\frac{d}{dt}\vec{x} = A\vec{x} + B\delta$$

(b) Now, assume for some specific component values we get the following differential equation:

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\vec{x}(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix}u(t). \tag{3}$$

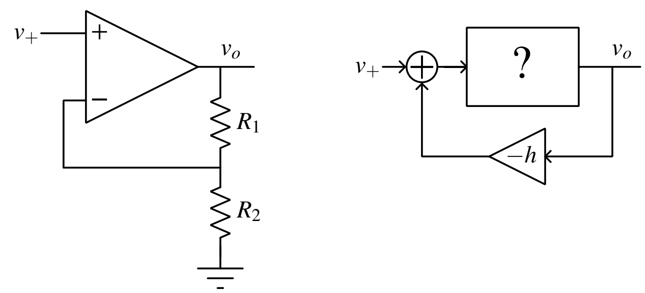
Unfortunately, we are unable to measure our state vector continuously. Suppose that we sample the system with some sampling interval T. Let us discretize the above system. Assume that we use piecewise constant voltage inputs $u(t) = u[k]$ for $t \in [kT, (k+1)T)$.

Calculate a discrete-time system for Equation (3)'s continuous-time vector system in the form:

$$\vec{x}[k+1] = A_d\vec{x}[k] + \vec{b}_d[k].$$

2. Feedback Control of Op-Amps (X pts)

You have seen op-amps in negative feedback many times, and you have learned about feedback control. You may not have realized it yet, but these are actually related to each other.



Here, we introduce a dynamic model for a non-ideal op-amp in negative feedback:

$$\frac{d}{dt}v_o(t) = -v_o(t) + Gu(t)$$

where v_o is the output voltage, and $u(t) = \big(v_+(t) - v_-(t)\big)$ where v_+ and v_- are the voltages at the positive and negative inputs respectively, and $G > 2$ is a parameter that defines the op-amp's behavior.

(a) (X pts) Given the dynamic model for the nonideal op-amp, **assuming v_+ and v_- are not changing, for what value of v_o will v_o not be changing?** Your answer should depend on v_+, v_-, G.

(b) (X pts) In the above, $v_-(t) = hv_o(t)$. **Pick values for the resistors R_1 and R_2 so that h equals $\frac{1}{2}$.**

(c) (8 pts) Suppose we place the nonideal op-amp in resistive negative feedback using a voltage divider whose ratio is $h = \frac{1}{2}$, in other words set

$$u(t) = v_+(t) - hv_o(t).$$

Write out the new differential equation that relates $v_o(t)$ to $v_+(t)$. Is this system stable? Briefly state why or why not.

(d) (4 pts) If we had swapped the roles of the positive and negative terminals of the op-amp (i.e. had hooked the nonideal op-amp up in positive feedback so that $u(t) = hv_o(t) - v_-(t)$.) **would the resulting closed-loop system have been stable? Briefly state why or why not.**

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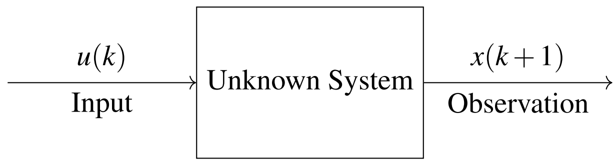
(e) (10 pts) For the closed-loop system in negative feedback (from part (c)) using resistor values that set $h = \frac{1}{2}$, assume that the output voltage starts at 0V at time 0 and that $v_+(t)$ was 0V but then jumps up to 1V at time 0. **How long will it take for $v_o(t)$ to reach 1V**? Your answer should be in terms of G.

(f) (4 pts) **What happens to the answer of the previous part if $G \to \infty$?**

3. System Identification

In this question, we will take a look at how to **identify** a system by taking experimental data taken from a (presumably) linear system to learn a discrete-time linear model for it using the least-squares.

The visual below shows our experimental procedure of giving an input sequence $u(k)$ and taking observations $x(k+1)$.



We will start by assuming the model for the system is $x(k+1) = \alpha x(k) + \beta u(k) + e(k)$.

(a) Given the sequence of inputs $u(0), u(1), u(2)$ and initial state $x(0)$ set up a Least Squares problem to estimate α and β.

(b) When will the Least Squares problem set up in part (a) have a unique solution?

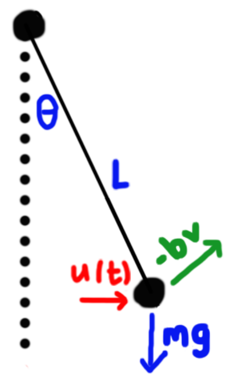
(c) Using this fact from the previous part, what is the minimum number of measurements we need to make in order to set up a Least Squares problem?

(d) Given the initial state $x(0) = 1$, input sequence $u(0) = 1, u(1) = 1$ and observations $x(1) = -1, x(2) = 3$, provide estimates for α and β.

(e) Now let's consider the model $x(k+1) = \alpha x(k) + \beta_0 + \beta_1 u(k) + e(k)$. Give an input sequence of length 3 where Least Squares will fail.

4. Inverted Pendulum

We will now take a look at the same pendulum from the previous question but around a new equilibrium point $\vec{x}^* = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$ and $u^* = 0$.



To recall, the kinetics of this pendulum over time can be represented by the following differential equation:

$$\frac{d^2\theta}{dt^2} + \frac{b}{m}\frac{d\theta}{dt} + \frac{g}{L}\sin\theta = \frac{\cos\theta}{mL}u \tag{4}$$

(a) **What is the linearized system around this operating point?**

(b) **Is this linearized system stable when** $b > 0$?

(c) **Show that the system can be put in CCF for the given physical values:** $b = 2,\ m = 2,\ g = 10,\ L = \frac{1}{2}$.
Hint: You will have to make some modification to the input.

(d) **Is this linearized system controllable?**

(e) **Design a feedback controller so that the eigenvalues of the system will be** $\lambda_1 = -2,\ \lambda_2 = -3.$

(f) **Suppose that we our feedback controller is limited and can only give feedback to the** θ **variable. In other words,** $K = \begin{bmatrix} k & 0 \end{bmatrix}$**. Is it still possible to pick eigenvalues that will make the system stable?**

5. Controllable Car Form

Recall the car-system can represent the system with the following differential equation with position $p(t)$ and velocity $v(t)$

$$\frac{d}{dt}\begin{bmatrix} p(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u(t) \tag{5}$$

Assuming $M = 1$ let us discretize this system with sampling rate $T = 1$.

$$\begin{bmatrix} p[n+1] \\ v[n+1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} p[n] \\ v[n] \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} u[n] \tag{6}$$

Since the discretized system is unstable, we will try to build a feedback controller $u[n] = -K\vec{x}[n]$ to stabilize its eigenvalues.

(a) **What is the closed loop matrix A_{cl}?**

(b) **Verify that this system is controllable.**

(c) **Find the transformation T that puts this system in Controllable Canonical Form**

(d) **Pick values for K_c so that places both eigenvalues at $\lambda = \frac{1}{2}$ in the controller basis.**

(e) **Find the controller $u[n] = -K\vec{x}[n]$ in the standard basis.**

6. Day at the Races

Forrest is building his SIXT33N Car to compete at the annual 16B racing competition. In this competition, cars can take any path, but they must stop exactly at the finish line.

Luckily Forrest has found a shortcut in the track so that the car can move in a straight line from start to finish. Therefore, he will use the following discrete-time model giving inputs to his car that weighs 10 kg every 1 s.

$$\begin{bmatrix} p[n+1] \\ v[n+1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p[n] \\ v[n] \end{bmatrix} + \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix} u[n] \tag{7}$$

(a) Assuming the car starts at rest, the finish line is 40 m away. **What inputs can Forrest give to reach the finish line in** 2 s?

(b) Upon giving these inputs, Forrest's car explodes due to the inputs being too strong. Therefore, he builds a new car and now tries to reach the finish line in 5 s.
Set up an optimization problem of the form

$$\min_{\vec{w} \in \mathbb{R}^5} \|\vec{w}\|^2 \text{ subject to } H\vec{w} = \vec{y} \tag{8}$$

so that the car can reach the finish line with minimum energy.

(c) Forrest creates 5 checkpoints $\vec{r}[1], \ldots, \vec{r}[5]$ to ensure that his car is moving in the right direction. **Set up an optimization problem of the form**

$$\min_{\vec{w} \in \mathbb{R}^{12}} \|\vec{w}\|^2 \text{ subject to } G\vec{w} = \vec{y} \tag{9}$$

that minimizes the both the energy and the squared distances from each checkpoint.
Hint: Define distances $\vec{d}[k] = \vec{x}[k] - \vec{r}[k]$ and write out the State-Space equations for $\vec{x}[k]$. Then try to create a vector $\vec{w} \in \mathbb{R}^{12}$ that represents both the distances and inputs.

(d) Forrest notices that Simon's car is sending out distrubances that prevents his car from moving straight. Therefore, Forrest sets up a new state-space model with $\vec{d}[n] = \vec{x}[n] - \vec{r}[n]$.

$$\vec{d}[n+1] = A\vec{d}[n] + B(u[n] - u^*[n])$$

where $u^*[n]$ are the control inputs from the previous part that reach the target with minimum energy. **Explain how Forrest can design a feedback controller to ensure that the car will be close to the checkpoints** $\vec{r}[n]$.

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