EECS 16B Designing Information Devices and Systems II
Fall 2019

Note j

1 What are Complex Numbers?

1.1 Introduction

Most students have some basic background in complex numbers ($\mathbb{C}$) from high school. The purpose of this note is to solidify this understanding. With that, let's begin with the most basic definition: $j = \sqrt{-1}$ (in most engineering discplines, we will use j, so as not confuse ourselves with current or the identity matrix I. Besides, numpy uses j as well). All complex numbers are of the form $z = a + jb$, where a is the **real part** and b is the **imaginary part**. This form is more commonly known as the **rectangular form**, and as we will see, addition in this form is very easy and akin to vector addition.

The **complex conjugate**[1] of z, represented by $\bar{z}$ (and sometimes by z^*), is defined as follows:

$$\bar{z} = \overline{(a + jb)} = a - jb \tag{1}$$

The **magnitude** of a complex number, z, is given by

$$|z| = \sqrt{a^2 + b^2} \tag{2}$$

and the **phase** is given by

$$\theta = \angle z = \text{atan2}(b, a) \tag{3}$$

Here, $\text{atan2}(y, x)$ is a function[2] that returns the angle from the positive x-axis to the vector from the origin to the point (x, y).

1.2 Basic Operations

Let's say we have two complex numbers $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$. As you may recall, addition is defined as follows:

$$z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2) \tag{4}$$

This should look very similar to the addition of vectors in 2D. Multiplication is a little more complicated, but it behaves very similar to polynomial multiplication, except the indeterminate (i.e. the variable) is replaced by j, and we have j^2 = -1:

$$\begin{aligned} z_1 \times z_2 &= (a_1 + jb_1) \times (a_2 + jb_2) && (5)\\ &= a_1 * a_2 + jb_1 * a_2 + ja_1 * b_2 + j^2 b_1 * b_2 && (6)\\ &= (a_1 * a_2 - b_1 * b_2) + j(a_1 * b_2 + a_2 * b_1) && (7) \end{aligned}$$

[1] $\bar{z}$ is also used in digital logic to represent 'NOT' operation

[2] See its relation to $\tan^{-1}\left(\frac{y}{x}\right)$ at `https://en.wikipedia.org/wiki/Atan2`.

Before we move on to division, let's see what the multiplicative inverse would look like:

$$\begin{aligned}\frac{1}{z} &= \frac{1}{a+jb} = \frac{1}{a+jb} \times \frac{a-jb}{a-jb} \\ &= \frac{a-jb}{a^2+b^2} = \frac{a}{a^2+b^2} - j\frac{b}{a^2+b^2} \end{aligned} \tag{8}$$

Above, we multiply both the numerator and denominator by $\bar{z}$. This allows us to make the denominator real, and we also observe $\bar{z} \times z = |z|^2$.

Following the same train of thought, let's define division as well:

$$\frac{z_1}{z_2} = \frac{a_1 + jb_1}{a_2 + jb_2} \tag{9}$$

$$= \frac{(a_1 + jb_1) \times (a_2 - jb_2)}{a_2^2 + b_2^2} \tag{10}$$

$$= \frac{(a_1 * a_2 + b_1 * b_2) - j(a_1 * b_2 - a_2 * b_1)}{a_2^2 + b_2^2} \tag{11}$$

At line (10), we subsititute the multiplicative inverse found in (8), and we continue by carrying out the multiplication as defined in (7).

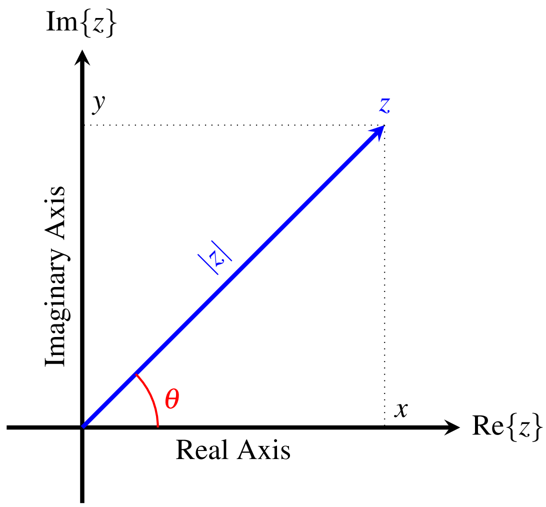

Figure 1: Complex number z depicted as a vector in the complex plane.

1.3 Complex Plane

The complex plane is an extension of the real number line; by adding another independent axis to represent the purely imaginary numbers (i.e. those with the real part equal to zero) we can represent all complex numbers. As shown in Figure 4, a complex number $z = x + jy$ has an intercept at x along the **real axis** and y along the **imaginary axis**. We can also see visually that the magnitude of z is its distance from the origin, and the phase is the angle from the positive real axis.

2 The Polar Form

2.1 Revisiting Multiplication

With our understanding of the complex plane, let's see how a complex number $z = 1 + j0$ changes as we multiply it with $z_1 = 1 + j$ repeatedly:

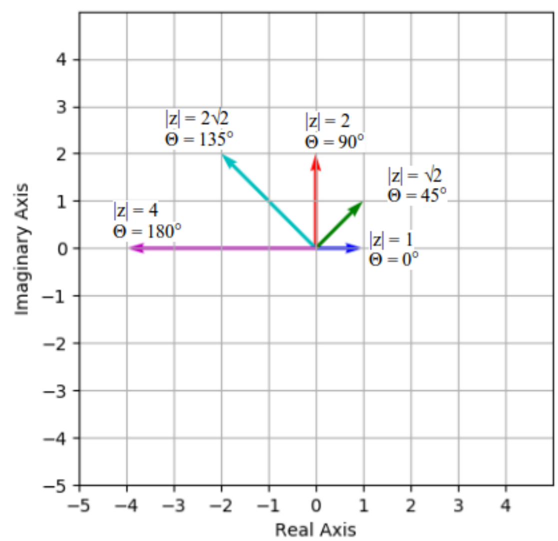

Figure 2: Multiplying z (in blue) by z_1 repeatedly

Clearly, it looks like z seems to be rotating by 45°. Is this an arbitrary angle? No, the angle of rotation seems to related to the phase $\angle z_1 = 45°$. So, can we think of some representation of complex numbers where this additive property of angles is natural?

How can we turn multiplication into some kind of addition? Well, we can represent the phase in an exponent. What if we used 'e' as the base for our exponent, and just set $z = e^{\angle z}$? Unfortunately, a pure real exponential will blow up as $\angle z$ increases, but this is not the behaviour we see with complex number angles. It is easy to see that any real number in the exponent couldn't possibly behave the right way — angles spin around while real exponents either go to zero or blow up. So if the thing in the exponent can't be real, for now, let's take a leap of faith and hope that something imaginary works! i.e. Multiply the thing in the exponent by j, i.e. $z = e^{j\angle z_2}$. We don't yet know how this will behave, but let's push forward.

The above motivation just captured the rotation property of multiplication by complex numbers. Futhermore, there is some scaling as well. More precisely, for the example of $1 + j$, it is scaling by $\sqrt{2}$. This cannot be a coincidence, so how can we express this scaling with our new form? We could multiply the magnitude of z_2 with the exponential, i.e. $z_2 = |z_2|e^{j\angle z_2}$.

Now, let's try to isolate the effect of rotation by multipling $z = 1 + j0$ by $z_2 = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$ (note that $|z_2| = 1$) repeatedly. We get the following plot:

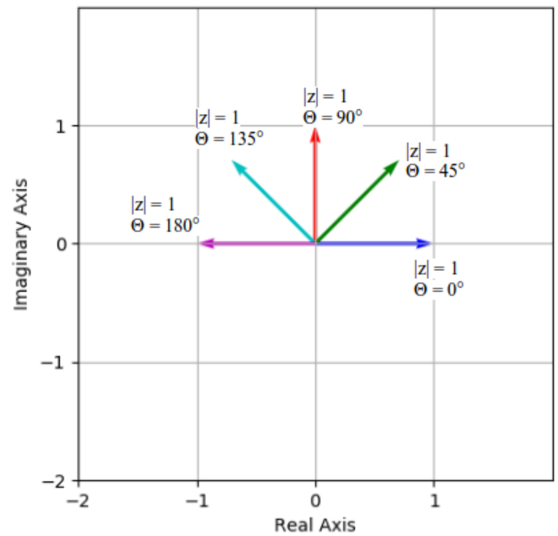

Figure 3: Multiplying z (in blue) by z_2 repeatedly

Clearly, we have a rotation by 45° as before, and we do not have any scaling. So, our intuition of multiplying the magnitude with the exponential agrees with this example as well. This is good news and increases our degree of comfort and confidence with our guess. But we really do need to justify it more thoroughly to be able to lean on it.

2.2 Developing the Polar Form and Euler's Equation

In the last section we conjectured,

$$z = |z|e^{j\angle z}. \tag{12}$$

But what does $e^{j\angle z}$ even mean? It is a natural first step, from our experiences, to use the exponential's definition in the form of its Taylor's expansion around 0 (or the Maclaurin's expansion of '$e^{\cdot}$'):

$$\begin{aligned}
z &= |z|e^{j\theta} && (13)\\
&= |z|\left(1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \cdots + \frac{(j\theta)^{2n}}{2n!} + \frac{(j\theta)^{2n+1}}{(2n+1)!} + \cdots\right) && (14)\\
&= |z|\left(1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \cdots + (-1)^n\frac{\theta^{2n}}{2n!} + j(-1)^n\frac{\theta^{2n+1}}{(2n+1)!} + \cdots\right) && (15)\\
&= |z|\left[\left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4} - \cdots + (-1)^n\frac{\theta^{2n}}{2n!} + \cdots\right) + j\left(\theta - \frac{\theta^3}{3} + \cdots + (-1)^n\frac{\theta^{2n+1}}{(2n+1)!} + \cdots\right)\right] && (16)\\
&= |z|\left(\cos(\theta) + j\sin(\theta)\right). && (17)
\end{aligned}$$

Here, $\theta = \angle z$ and the angles are clearly measured in radians. At (15), we used the fact that j^k has a regular periodically repeating pattern to it: $j^0 = +1$, $j^1 = +j$, $j^2 = -1$, $j^3 = -j$, $j^4 = +1$, and so it goes. Basically we just need to look at the remainder[3] that we get after we divide k by 4. If that remainder is 0 (i.e. k is a

[3] Such remainder operations are often called mod operations, and they play a major role in our follow-on course 70. But in 16B, you will begin to be prepared for thinking about such cyclic behavior because the complex numbers exhibit it very naturally.

multiple of 4 or alternatively $k = 4\ell$ for some integer ℓ), then $j^k = +1$. If that remainder is 1 (i.e. k is 1 plus a multiple of 4 or alternatively $k = 4\ell + 1$ for some integer ℓ), then $j^k = +j$. If that remainder is 2 (i.e. k is 2 plus a multiple of 4 or alternatively $k = 4\ell + 2$ for some integer ℓ), then $j^k = -1$. If that remainder is 3 (i.e. k is 3 plus a multiple of 4 or alternatively $k = 4\ell + 3$ for some integer ℓ), then $j^k = -j$. Going from equation (16) to (17), we recognize and substitute for the Taylor expansions of sine and cosine around at 0. Let's analyze our result in equation (17) with reference to the diagram of the complex plane below:

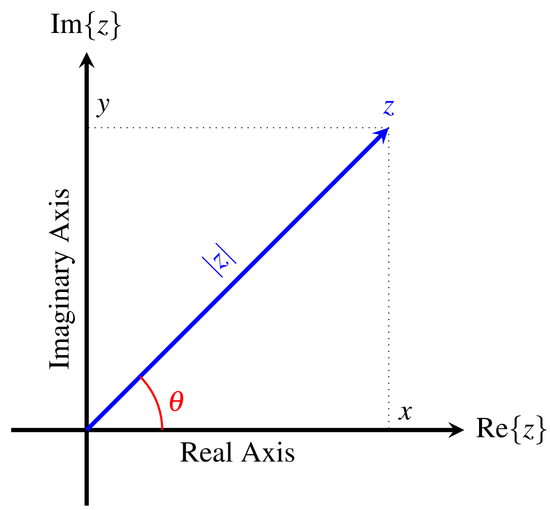

Figure 4: Complex number z depicted as a vector in the complex plane.

Recalling basic definitions from trigonometry, we can see that $x = |z|\cos(\theta)$ and $y = |z|\sin(\theta)$, hence $z = x + jy = \cos(\theta) + j\sin(\theta)$, agreeing with our previous result. Hence, the form we guessed in the earlier section was correct and we have come back full circle, connecting with the rectangular form we discussed in Section 1.

Concept Check: Using Euler's equation:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{18}$$

write sine and cosine as sums of complex exponentials.
Solution:

$$e^{-j\theta} = \cos(-\theta) + j\sin(-\theta) = \cos(\theta) - j\sin(\theta) \tag{19}$$

$$\text{Adding and Subtracting equation (18) and (19), we get :}$$

$$2\cos(\theta) = e^{j\theta} + e^{-j\theta} \Rightarrow \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{20}$$

$$2j\sin(\theta) = e^{j\theta} - e^{-j\theta} \Rightarrow \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} = \frac{-je^{j\theta} + je^{-j\theta}}{2} \tag{21}$$

This is the source of the famous identity $e^{j\pi} + 1 = 0$, connecting five very fundamental numbers together: 0 (the additive identity), 1 (the multiplicative identity), e (the base of the natural logarithm, defined because we want a function whose derivative was itself), j (the basic imaginary number $\sqrt{-1}$), and π (the area of a perfect circle with radius 1). Such remarkable beauty is what marks the subject of complex analysis[4] more generally.

[4]The historical tradition within Electrical Engineering has traditionally placed a lot of emphasis on complex analysis alongside

2.3 Conclusion

The polar form of z is a very important representation as it greatly simplifies multiplication. The polar form of $z = a + jb$ is given as follows:

$$z = |z|e^{j\angle z}$$

The conjugate of z is given as $\overline{z} = |z|e^{-j\angle z}$ since $\sin(-\theta) = -\sin(\theta)$ while $\cos(-\theta) = \cos(\theta)$ from trigonometry.

Multiplication in this form is given as:

$$z_1 \times z_2 = (|z_1|e^{j\angle z_1}) \times (|z_2|e^{\angle z_2}) = (|z_1| * |z_2|)e^{j(\angle z_1 + \angle z_2)}$$

If we look carefully, we can realize that complex multiplication is nothing but a rotation operation, followed by scaling. We are essentially rotating z_1 by $\angle z_2$ in the counter-clockwise direction[5], and scaling it by a factor of $|z_2|$. Of course, this conclusion would have been difficult to make precise without the use of polar forms. We will solidfy this view of rotations further in the next section where we will model complex numbers as matrices to augment the vector intuition we have gained thus far.

2.4 Useful Identities

Complex Number Properties

Rectangular vs. polar forms: $z = x + jy = |z|e^{j\theta}$

where $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$, $\theta = \text{atan2}(y, x)$. We can also write $x = |z|\cos\theta$, $y = |z|\sin\theta$.

Euler's identity: $e^{j\theta} = \cos\theta + j\sin\theta$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}, \ \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Complex conjugate: $\overline{z} = x - jy = |z|e^{-j\theta}$

$$\overline{(z+w)} = \overline{z} + \overline{w}, \ \overline{(z-w)} = \overline{z} - \overline{w}$$

$$\overline{(zw)} = \overline{z}\,\overline{w}, \ \overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$$

$\overline{z} = z \Leftrightarrow z$ is real

$\overline{z} = -z \Leftrightarrow z$ is purely complex, i.e. no real part

$$\overline{(z^n)} = (\overline{z})^n$$

Complex Algebra

Let $z_1 = x_1 + jy_1 = |z_1|e^{j\theta_1}$, $z_2 = x_2 + jy_2 = |z_2|e^{j\theta_2}$.

(Note: we adopt the easier representation between rectangular form and polar form.)

Addition: $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

Multiplication: $z_1 z_2 = |z_1||z_2|e^{j(\theta_1 + \theta_2)}$

Division: $\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}e^{j(\theta_1 - \theta_2)}$

Power: $z_1^n = |z_1|^n e^{jn\theta_1}$

$z_1^{\frac{1}{2}} = \pm|z_1|^{\frac{1}{2}} e^{j\frac{\theta_1}{2}}$

(Note: Just like square roots are not unique, other fractional powers of z_1 are not unique as well)

Useful Relations

$$-1 = j^2 = e^{j\pi} = e^{-j\pi}$$

$$j = e^{j\frac{\pi}{2}} = \sqrt{-1}$$

$$-j = -e^{j\frac{\pi}{2}} = e^{-j\frac{\pi}{2}}$$

$$\sqrt{j} = (e^{j\frac{\pi}{2}})^{\frac{1}{2}} = \pm e^{j\frac{\pi}{4}} = \frac{\pm(1+j)}{\sqrt{2}}$$

real analysis and linear algebra. You will see some evidence of this in later courses like 120 and 128, and even in courses like 105, 140, and 142. However, the 16AB sequence is almost completely rooted in linear algebra because linear algebra is far simpler, and while sometimes lacking the delicate beauty of complex analysis, it can be more robust. At advanced levels, you need to know both. There is a reason why Complex Analysis 185 is considered an absolutely core course for Math majors and minors, along with Real Analysis 104, Linear Algebra 110, and Abstract Algebra 113. In the EECS lower division, 16AB hit basic linear algebra, 16B touches on some ideas from real analysis and only skirts vague hints of complex analysis, and 70 touches on some ideas from abstract algebra. We encourage our students to take the core upper division Math courses for a deeper understanding and appreciation for these foundational conceptual tools.

[5] It is common to define the counter-clockwise direction as the positive direction while measuring angles in the Cartesian plane.

Concept Check: Verify the above identities for yourself if you have not done so in prior classes.

3 Complex Numbers modeled using Matrices

Viewing complex numbers as vectors definitely seems attractive and it does fit into our visualization of the complex plane, but it has a major flaw — vectors do not naturally multiply, but complex numbers do. In fact, multiplication is the *raison d'etre* for complex numbers. So, how do we get a better model? What both adds and multiplies? Enter matrices, and more specifically scaled rotation matrices.

3.1 Matrix form of rotations

But first, what is a rotation matrix? To begin answering this question, we need to first understand what a rotation tranformation would look like. Rotating the vector $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by angle θ in the counter clockwise direction would give us $\widetilde{\vec{e}_1} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$, and similarly for $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we will get $\widetilde{\vec{e}_2} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$. Hence, we can describe the rotation transform (by angle θ) as the following matrix:

$$\widetilde{\vec{v}} = R_\theta \vec{v} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \vec{v} \tag{22}$$

An important thing to note is this rotation matrix has orthonormal columns[6]. Next, what would happen if we rotated a vector by θ_1 and then by θ_2? Well, it would be equivalent to rotating it by $\theta_1 + \theta_2$, hence we have:

$$R_{\theta_1} * R_{\theta_2} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix} = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} = R_{\theta_1+\theta_2} \tag{23}$$

Concept Check: Use basic trigonometry (in particular, the sum-angle formulas for sine and cosine that you probably derived in high school) to check the equality established in equation (23). Futhermore, rotations in 2D are commutative[7] as well. Show that this is true by proving $R_{\theta_1} * R_{\theta_2} = R_{\theta_2} * R_{\theta_1}$

When we look back at rotation matrix in (22), it bears some resemblance to the Euler form (equation 17) we discovered in the previous section. If we have a complex number $z = a + jb = \cos(\theta) + j\sin(\theta)$, where $|z| = 1$ (for simplicity, we will look at scaling a bit later) and $\angle z = \theta$, then we could define a matrix $Z_{(a,b)}$ as follows:

$$Z_{(a,b)} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \tag{24}$$

Concept Check: Check that this matrix has orthogonal columns.

[6]The columns in a matrix with orthonormal columns all have norm 1 and are mutually orthogonal to each other (i.e. their inner products with each other are zero). Such matrices are commonly referred to as **orthogonal** matrices in the mathematical literature.

[7]This commutative property for rotations only holds for 2D spaces, and not for 3D spaces. Take a second to think about this!

We can express the fundamentally two-dimensonal nature of such matrices by expressing them using a clear basis:

$$Z_{(a,b)} = aI + bJ \text{ where } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{25}$$

Notice that $J^2 = -I$ above, and so the matrix J acts like the counterpart of the basic imaginary number j.

What is a complex conjugate in this representation? What can we do to swap the b and $-b$ in the matrix above? Indeed we see that **transposing** the matrix corresponds to complex conjugation of the underlying complex number. It has no effect on a scaled identity matrix which would correspond to a purely real number. But $J^T = -J$.

Next, let's look at the scaling. In this case, we have $z = a + jb$, with $|z| = \sqrt{a^2+b^2}$. To account for this in our matrix model, we can factor out $|z|$ as follows:

$$Z_{a,b} = \sqrt{a^2+b^2} \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} & -\frac{b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{bmatrix} \tag{26}$$

Looking at the above form, the factor out in the front is responsible for the scaling. From the figure below, we can find $\theta = \text{atan2}(b,a)$ such that $\cos(\theta) = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin(\theta) = \frac{b}{\sqrt{a^2+b^2}}$.

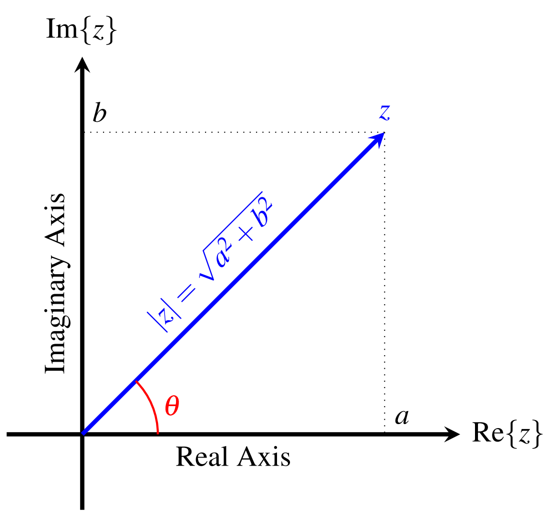

Figure 5: Complex number $z = a + jb$ represented as a vector in the complex plane.

Now, let's see if this model fits with everything that we know about complex arithmetic.

3.1.1 Addition:

For two complex numbers, $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$, we have:

$$Z_{(a_1,b_1)} + Z_{(a_2,b_2)} = \begin{bmatrix} a_1 + a_2 & -(b_1+b_2) \\ b_1 + b_2 & a_1 + a_2 \end{bmatrix} = Z_{(a_1+a_2, b_1+b_2)}$$

Hence, it satisfies our definition of addition.

3.1.2 Multiplication by real number:

Let $z = a + jb$, then $\lambda z = \lambda a + j\lambda b$, where λ is a real number. This can be easily extended to our matrix form as well:

$$\lambda Z_{(a,b)} = \lambda \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \lambda a & -\lambda b \\ \lambda b & \lambda a \end{bmatrix} = Z_{(\lambda a, \lambda b)}$$

3.1.3 Multiplication by a complex number:

Finally, and the reason we are pursuing this representation, multiplication by another complex number. Let $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$, then we have $z_1 \times z_2 = (a_1 * a_2 - b_1 * b_2) + j(a_1 * b_2 + a_2 * b_1)$. Let's check if this is the case with matrix multiplication:

$$Z_{(a_1,b_1)} * Z_{(a_2,b_2)} = \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1 * a_2 - b_1 * b_2 & -(a_1 * b_2 + a_2 * b_1) \\ a_1 * b_2 + a_2 * b_1 & a_1 * a_2 - b_1 * b_2 \end{bmatrix} = Z_{(a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1)}$$

Note that since our 2D rotations are communtative, so are the multiplications with complex numbers.

Multiplication by the complex conjugate also clearly gives an identity matrix with the magnitude squared along the diagonal.

It turns out, although we will not show this here, that even the natural generalization of exponentiation to matrices works with this matrix model for complex numbers. We get $e^{a+jb} = e^a e^{jb} = e^a(\cos b + j \sin b)$. Actually, the understanding of the natural generalization of exponentiation to matrices requires understanding the solutions to systems of differential equations, where the complex exponentiation case turns out to represent the behavior of RLC circuits. To understand this requires understanding the eigenvalues of the kinds of matrices we find here, but that is a subject of a different note.

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