EE16B - Spring'20 - Lecture 8A Notes[1]

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Discretization and Controllability

Discretization for Vector State Models

In the last lecture we considered the linear continuous-time system

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t), \tag{1}$$

where $\vec{x}(t)$ is sampled every T units of time, leading to the sequence

$$\vec{x}_d(k) := \vec{x}(kT), \quad k = 0, 1, 2, \ldots \tag{2}$$

If $\vec{u}(t)$ is constant between the samples:

$$\vec{u}(t) = \vec{u}_d(k) \quad t \in [kT, (k+1)T), \tag{3}$$

then we can derive a discrete-time model

$$\vec{x}_d(k+1) = A_d\vec{x}_d(k) + B_d\vec{u}_d(k) \tag{4}$$

that describes how the state of the continuous-time system evolves from one sample to the next.

Last time we did this derivation for the scalar system

$$\frac{d}{dt}x(t) = \lambda x(t) + bu(t), \tag{5}$$

and obtained

$$x_d(k+1) = \lambda_d x_d(k) + b_d u_d(k) \tag{6}$$

where

$$\lambda_d = e^{\lambda T}, \quad b_d = b\int_0^T e^{\lambda s} ds = \begin{cases} bT & \text{if } \lambda = 0 \\ b\frac{e^{\lambda T}-1}{\lambda} & \text{if } \lambda \neq 0. \end{cases} \tag{7}$$

To generalize this result to the vector state model (1) let's first assume A is diagonal and B is a column vector:

$$A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

Then (1) consists of decoupled scalar equations

$$\frac{d}{dt}x_i(t) = \lambda_i x_i(t) + b_i u(t)$$

and we can discretize each as in (6)-(7). We then assemble the discretized scalar equations into the vector form (4) with

$$A_d = \begin{bmatrix} e^{\lambda_1 T} & & \\ & \ddots & \\ & & e^{\lambda_n T} \end{bmatrix}, \quad B_d = \begin{bmatrix} \int_0^T e^{\lambda_1 s} ds & & \\ & \ddots & \\ & & \int_0^T e^{\lambda_n s} ds \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

Next suppose A is not diagonal, but *diagonalizable*[2]; that is, it has linearly independent eigenvectors $\vec{v}_1, \cdots, \vec{v}_n$. Then $V = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$ is invertible and, as we saw last time, the change of variables

$$\vec{z} = V^{-1}\vec{x}$$

results in the new state equations

$$\frac{d}{dt}\vec{z}(t) = \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{A_{\text{new}}} \vec{z}(t) + \underbrace{V^{-1}B}_{B_{\text{new}}} u(t).$$

Since A_{new} is diagonal we apply the result above for the diagonal case and obtain

$$\vec{z}_d(k+1) = \begin{bmatrix} e^{\lambda_1 T} & & \\ & \ddots & \\ & & e^{\lambda_n T} \end{bmatrix} \vec{z}_d(k) + \begin{bmatrix} \int_0^T e^{\lambda_1 s} ds & & \\ & \ddots & \\ & & \int_0^T e^{\lambda_n s} ds \end{bmatrix} V^{-1} B u_d(k).$$

To return to the original state variables, note that

$$\vec{x}_d(k) = V\vec{z}_d(k), \quad \vec{z}_d(k) = V^{-1}\vec{x}_d(k),$$

and, therefore,

$$\begin{aligned} \vec{x}_d(k+1) &= V\vec{z}_d(k+1) \\ &= V\left(\begin{bmatrix} e^{\lambda_1 T} & & \\ & \ddots & \\ & & e^{\lambda_n T} \end{bmatrix} \vec{z}_d(k) + \begin{bmatrix} \int_0^T e^{\lambda_1 s} ds & & \\ & \ddots & \\ & & \int_0^T e^{\lambda_n s} ds \end{bmatrix} V^{-1} B u_d(k) \right) \\ &= \underbrace{V \begin{bmatrix} e^{\lambda_1 T} & & \\ & \ddots & \\ & & e^{\lambda_n T} \end{bmatrix} V^{-1}}_{=A_d} \vec{x}_d(k) + \underbrace{V \begin{bmatrix} \int_0^T e^{\lambda_1 s} ds & & \\ & \ddots & \\ & & \int_0^T e^{\lambda_n s} ds \end{bmatrix} V^{-1} B}_{=B_d} u_d(k). \qquad (8) \end{aligned}$$

Summary: If A in (1) has linearly independent eigenvectors $\vec{v}_1, \cdots, \vec{v}_n$ with corresponding eigenvalues $\lambda_1, \cdots, \lambda_n$, then we form the invertible matrix $V = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$ and obtain the discretized model (4) where A_d and B_d are as in (8).

[2] Recall that A is diagonalizable if it has distinct eigenvalues. If there are repeated eigenvalues A may or may not be diagonalizable: we need to check whether it has n linearly independent eigenvectors or not.

<u>Example 1</u>: Consider the system (1) with

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and no input. The LC circuit model studied in Lecture 4A with $L = 1$, $C = 1$ had this form. As shown then, the eigenvalues/vectors are

$$\lambda_1 = j, \quad \lambda_2 = -j, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -j \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ j \end{bmatrix}.$$

Thus,

$$V = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \quad \text{and} \quad V^{-1} = \frac{1}{2j} \begin{bmatrix} j & -1 \\ j & 1 \end{bmatrix}.$$

Then, from (8),

$$\begin{aligned}
A_d &= V \begin{bmatrix} e^{\lambda_1 T} & \\ & e^{\lambda_2 T} \end{bmatrix} V^{-1} = V \begin{bmatrix} e^{jT} & \\ & e^{-jT} \end{bmatrix} V^{-1} \\
&= \frac{1}{2j} \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \begin{bmatrix} e^{jT} & \\ & e^{-jT} \end{bmatrix} \begin{bmatrix} j & -1 \\ j & 1 \end{bmatrix} \\
&= \frac{1}{2j} \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \begin{bmatrix} je^{jT} & -e^{jT} \\ je^{-jT} & e^{-jT} \end{bmatrix} \\
&= \frac{1}{2j} \begin{bmatrix} j(e^{jT} + e^{-jT}) & -(e^{jT} - e^{-jT}) \\ -j^2(e^{jT} - e^{-jT}) & j(e^{jT} + e^{-jT}) \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}(e^{jT} + e^{-jT}) & -\frac{1}{2j}(e^{jT} - e^{-jT}) \\ \frac{1}{2j}(e^{jT} - e^{-jT}) & \frac{1}{2}(e^{jT} + e^{-jT}) \end{bmatrix} \\
&= \begin{bmatrix} \cos T & -\sin T \\ \sin T & \cos T \end{bmatrix}.
\end{aligned}$$

Controllability

The solution of the discrete-time state model

$$\vec{x}(t+1) = A\vec{x}(t) + B\vec{u}(t), \tag{9}$$

where $\vec{x}(t)$ is an n-dimensional vector, can be obtained recursively as:

$$\begin{aligned}
\vec{x}(1) &= A\vec{x}(0) + B\vec{u}(0) \\
\vec{x}(2) &= A\vec{x}(1) + B\vec{u}(1) = A(A\vec{x}(0) + B\vec{u}(0)) + B\vec{u}(1) \\
&= A^2\vec{x}(0) + AB\vec{u}(0) + B\vec{u}(1) \\
\vec{x}(3) &= A\vec{x}(2) + B\vec{u}(2) = A(A^2\vec{x}(0) + AB\vec{u}(0) + B\vec{u}(1)) + B\vec{u}(2) \\
&= A^3\vec{x}(0) + A^2B\vec{u}(0) + AB\vec{u}(1) + B\vec{u}(2) \\
&\vdots \\
\vec{x}(t) &= A^t\vec{x}(0) + A^{t-1}B\vec{u}(0) + A^{t-2}B\vec{u}(1) + \cdots + AB\vec{u}(t-2) + B\vec{u}(t-1)
\end{aligned}$$

or, equivalently,

$$\vec{x}(t) = A^t\vec{x}(0) + \begin{bmatrix} B & AB & \cdots & A^{t-2}B & A^{t-1}B \end{bmatrix} \begin{bmatrix} \vec{u}(t-1) \\ \vec{u}(t-2) \\ \vdots \\ \vec{u}(1) \\ \vec{u}(0) \end{bmatrix}. \quad (10)$$

Can we find an input sequence $\vec{u}(0), \vec{u}(1), \ldots, \vec{u}(t-1)$ that brings the state from $\vec{x}(0)$ to any desired value $\vec{x}(t) = \vec{x}_{\text{target}}$ at some time t? If the answer is yes for any $\vec{x}_{\text{target}} \in \mathbb{R}^n$, the system is called *controllable*. Otherwise, the system is called *uncontrollable*. More precisely:

Definition. If, for every $\vec{x}_{\text{target}} \in \mathbb{R}^n$, there exist a t and an input sequence $\vec{u}(0), \vec{u}(1), \ldots, \vec{u}(t-1)$ such that $x(t) = \vec{x}_{\text{target}}$, then the system is *controllable*. If, for some $\vec{x}_{\text{target}} \in \mathbb{R}^n$, there exist no t and no input sequence $\vec{u}(0), \vec{u}(1), \ldots, \vec{u}(t-1)$ such that $x(t) = \vec{x}_{\text{target}}$, then the system is *uncontrollable*.

To investigate controllability further we assume the system has a single input, that is B is a column vector $\vec{b} \in \mathbb{R}^n$, and rewrite (10) as

$$\vec{x}(t) - A^t\vec{x}(0) = \begin{bmatrix} \vec{b} & A\vec{b} & \cdots & A^{t-2}\vec{b} & A^{t-1}\vec{b} \end{bmatrix} \begin{bmatrix} u(t-1) \\ u(t-2) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix}. \quad (11)$$

Achieving $x(t) = \vec{x}_{\text{target}}$ means making the left hand side equal to $\vec{x}_{\text{target}} - A^t\vec{x}(0)$. Thus, the system is controllable if we can arbitrarily assign the the left hand side to any desired vector in $\mathbb{R}^n$ with an appropriate choice of t and input sequence $u(0), u(1), \ldots, u(t-1)$.

This means that the system is controllable if the column space of

$$\begin{bmatrix} \vec{b} & A\vec{b} & \cdots & A^{t-2}\vec{b} & A^{t-1}\vec{b} \end{bmatrix}, \quad (12)$$

that is $\text{span}\{\vec{b}, A\vec{b}, \cdots, A^{t-2}\vec{b}, A^{t-1}\vec{b}\}$, is $\mathbb{R}^n$ for some t.

Note that (12) has t columns. Since we can't span $\mathbb{R}^n$ with fewer than n columns, we must try $t = n$ or higher to check whether the span is $\mathbb{R}^n$. However, as we will prove later, if the n columns

$$\vec{b}, A\vec{b}, \cdots, A^{n-2}\vec{b}, A^{n-1}\vec{b}$$

do not already span $\mathbb{R}^n$, adding more columns $A^n\vec{b}, A^{n+1}\vec{b}, \cdots$ will not enlarge the span to $\mathbb{R}^n$. This leads to the following conclusion:

$$\text{Controllability} \quad \Leftrightarrow \quad \text{span}\{\vec{b}, A\vec{b}, \cdots, A^{n-2}\vec{b}, A^{n-1}\vec{b}\} = \mathbb{R}^n.$$

We will further discuss this condition and its proof in the next lecture; for now we illustrate it with two examples.

Example 2: The system

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{b}} u(t),$$

where $n = 2$, is controllable because

$$\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad A\vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

are linearly independent and together span $\mathbb{R}^2$. If we wish to reach $\vec{x}_{\text{target}}$ from $\vec{x}(0)$ we can do so in $t = 2$ steps by solving

$$\vec{x}_{\text{target}} - A^2\vec{x}(0) = \begin{bmatrix} \vec{b} & A\vec{b} \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$$

for $u(0)$ and $u(1)$:

$$\begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^{-1} (\vec{x}_{\text{target}} - A^2\vec{x}(0))$$

Example 3: The system

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{b}} u(t),$$

where only $\vec{b}$ is different from Example 2, is *uncontrollable* because

$$A\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which is the same as $\vec{b}$, therefore $\text{span}\{\vec{b}, A\vec{b}\} \neq \mathbb{R}^2$. You can see that adding $A^2\vec{b}, A^3\vec{b}, \cdots$ does not enlarge the span, because all of these vectors are the same as $\vec{b}$.

The reason for uncontrollability becomes clear if we write the equation for the second state variable $x_2(t)$ explicitly:

$$x_2(t+1) = 2x_2(t).$$

The right hand side doesn't depend on $u(t)$ or $x_1(t)$, which means that $x_2(t)$ evolves independently and can be influenced neither directly by input $u(t)$, nor indirectly through the other state $x_1(t)$.