This homework is due on Wednesday, March 11, 2020, at 11:59PM.

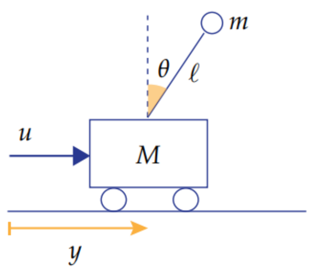
Self-grades are due on Monday, March 16, 2020, at 11:59PM.

1 Inverted Pendulum on a Rolling Cart (Mechanical)

Consider the inverted pendulum depicted below, which is placed on a rolling cart and whose equations of motion are given by:

$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2\ell\sin\theta - g\sin\theta\cos\theta\right)$$
$$\ddot{\theta} = \frac{1}{\ell(\frac{M}{m} + \sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2\ell\cos\theta\sin\theta + \frac{M+m}{m}g\sin\theta\right).$$

where we use $\dot{x}$ to denote the time derivative of x; that is, $\dot{y} = \frac{dy}{dt}$, $\dot{\theta} = \frac{d\theta}{dt}$, $\ddot{y} = \frac{d^2y}{dt^2}$ and $\ddot{\theta} = \frac{d^2\theta}{dt^2}$.



The problems below will prepare us for a future homework problem where we will design a control algorithm to stabilize the upright position.

a) Write the state model using the variables $x_1(t) = \theta(t)$, $x_2(t) = \dot{\theta}(t)$, and $x_3(t) = \dot{y}(t)$. We do not include $y(t)$ as a state variable because we are interested in stabilizing at the point $\theta = 0$, $\dot{\theta} = 0$, $\dot{y} = 0$, and we are not concerned about the final value of the position $y(t)$.

Solution

We have

$$\begin{aligned}
\dot{x_1} &= x_2 && \triangleq f_1(x_1, x_2, x_3, u)\\
\dot{x_2} &= \left(\frac{1}{l(\frac{M}{m}+\sin^2(x_1))}\right)\left(-\frac{u}{m}\cos(x_1) - x_2^2 l\cos(x_1)\sin(x_1) + \frac{M+m}{m}g\sin(x_1)\right) && \triangleq f_2(x_1, x_2, x_3, u)\\
\dot{x_3} &= \left(\frac{1}{\frac{M}{m}+\sin^2(x_1)}\right)\left(\frac{u}{m} + x_2^2 l\sin(x_1) - g\sin(x_1)\cos(x_1)\right) && \triangleq f_3(x_1, x_2, x_3, u)
\end{aligned}$$

b) Show that $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ is an equilibrium point with $u = 0$.

Solution

We check that $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$ with at the given point:

$$\begin{aligned}
\dot{x}_1 &= x_2 = 0 \\
\dot{x}_2 &= \left(\frac{1}{l\left(\frac{M}{m} + \sin^2(0)\right)}\right)\left(-\frac{0}{m}\cos(0) - 0 + \frac{M+m}{m} g \sin(0)\right) \\
&= \left(\frac{1}{l\left(\frac{M}{m}\right)}\right)(0 - 0 + 0) = 0 \\
\dot{x}_3 &= \left(\frac{1}{\frac{M}{m} + \sin^2(0)}\right)\left(\frac{0}{m} + 0^2 - g\sin(0)\cos(0)\right) = 0.
\end{aligned}$$

c) Linearize this model at the equilibrium $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, and $u = 0$, and indicate the resulting A and B matrices.

Solution

Note that

$$\begin{array}{lll}
\frac{\partial f_1}{\partial x_1}(0,0,0,0) = 0 & \frac{\partial f_1}{\partial x_2}(0,0,0,0) = 1 & \frac{\partial f_1}{\partial x_3}(0,0,0,0) = 0 \\
\frac{\partial f_2}{\partial x_1}(0,0,0,0) = \frac{M+m}{lM} g & \frac{\partial f_2}{\partial x_2}(0,0,0,0) = 0 & \frac{\partial f_2}{\partial x_3}(0,0,0,0) = 0 \\
\frac{\partial f_3}{\partial x_1}(0,0,0,0) = -\frac{m}{M} g & \frac{\partial f_3}{\partial x_2}(0,0,0,0) = 0 & \frac{\partial f_3}{\partial x_3}(0,0,0,0) = 0,
\end{array}$$

and,

$$\frac{\partial f_1}{\partial u}(0,0,0,0) = 0 \quad \frac{\partial f_2}{\partial u}(0,0,0,0) = -\frac{1}{lM} \quad \frac{\partial f_3}{\partial u}(0,0,0,0) = \frac{1}{M}.$$

Since $x^* = 0$ and $u^* = 0$, we can use the same state variables x and u, instead of $\tilde{x}$ and $\tilde{u}$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \approx \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ \frac{M+m}{lM} g & 0 & 0 \\ -\frac{m}{M} g & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -\frac{1}{lM} \\ \frac{1}{M} \end{bmatrix}}_{B} u.$$

2 Single-dimensional linearization

This is an exercise in linearizing a scalar system. The scalar nonlinear differential equation we have is

$$\frac{d}{dt} x(t) = \sin\big(x(t)\big) + u(t). \tag{1}$$

a) Find the equilibrium points for $u^* = 0$. You can do this by sketching $\sin(x)$ for $-4\pi \leq x \leq 4\pi$ and intersecting it with the horizontal line at 0. This will give you the equilibrium points x^* where $sin(x^*) + u^* = 0$.

Solution

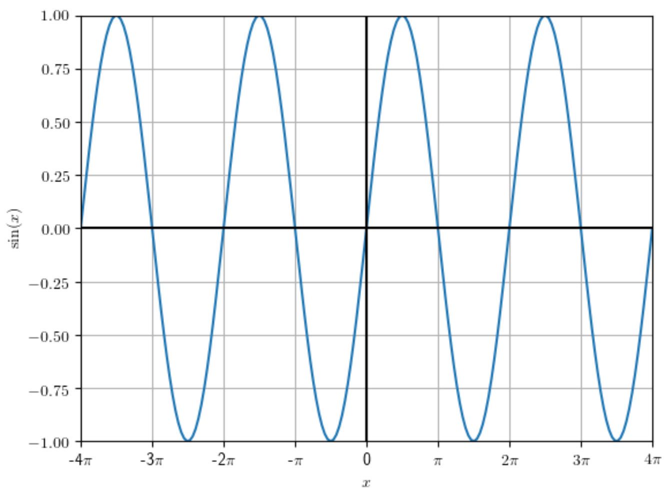


Figure 1: Plot of $\sin(x)$

We can see that all the multiples of π are where the line intersects the sine wave. This means that $x^*_m = m\pi$ is an equilibrium poin for any integer m.

b) Linearize the system (1) around the equilibrium $(x^*_0, u^*) = (0, 0)$. **What is the resulting linearized scalar differential equation for $\tilde{x}(t) = x(t) - x^*_0 = x(t) - 0$, involving $\tilde{u}(t) = u(t) - u^* = u(t) - 0$?**

Solution

First substituting for x and u in (1), then substituting the linearization for $\sin(x)$, we get,

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \sin\big(x(t)\big) + u(t) \\
\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} &= \sin\big(\tilde{x}(t)\big) + \tilde{u}(t) \\
&\approx \sin\big(x_0^*\big) + \tilde{x}(t)\frac{d}{dx}\sin(x)\Big|_{x=x_0^*} + \tilde{u}(t) \\
&= 0 + \tilde{x}(t)\cos(0) + \tilde{u}(t) \\
&= \tilde{x}(t) + \tilde{u}(t)
\end{aligned}$$

Notice that this approximate equality has been made an exact equality by calling the approximation error a disturbance $w(t)$.

c) For the linearized approximate system model that you found in the previous part, what happens if we try to discretize time to intervals of duration T? Assume now we use a piecewise constant control input over duration T, that T is small relative to the ranges of controls applied, and that we sample the state x every T (that is, at every $t = nT$, where n is an integer) as well. **Write out the resulting scalar discrete-time control system model.** This model is an approximation of what will happen if we actually applied a piecewise constant control input to the original nonlinear differential equation.

Solution

Let's look at the time interval $t \in [kT, (k+1)T]$. In this interval, we have constant input $u = u[k]$. Hence, we can rewrite the differential equation as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(t) + u[k]. \tag{2}$$

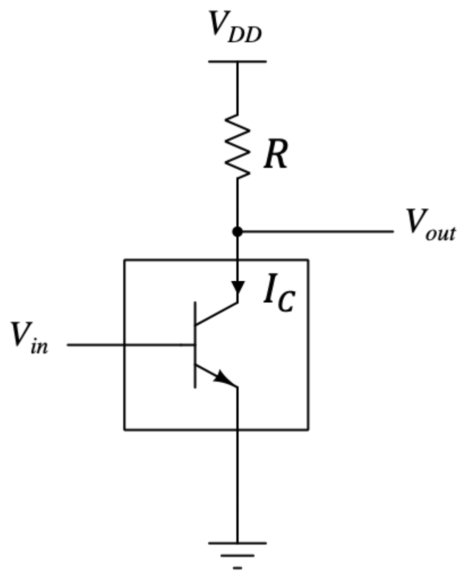
Following class notes

$$x(k+1) = e^T x[k] + u(k)(e^T - 1)$$

3 Linearizing for understanding amplification

Linearization isn't just something that is important for control, robotics, machine learning, and optimization — it is one of the standard tools for circuits

The circuit below is a voltage amplifier, where the element inside the box is a bipolar junction transistor (BJT).



The bipolar transistor in the circuit can be modeled quite accurately as a nonlinear, voltage-controlled current source, where the collector current I_C is given by

$$I_C(V_{in}) = I_S e^{\frac{V_{in}}{V_{TH}}} \tag{3}$$

where V_{TH} is the thermal voltage. We can assume $V_{TH} = 26$ mV at temperatures of 300K (close to room temperature). I_S is a constant whose exact value we are not giving you because we want you to find ways of eliminating it in favor of other quantities whenever possible.

With this amplifier, small variations in the input voltage V_{in} can turn into large variations in the output voltage V_{out} under the right conditions. We're going to investigate this amplification using linearization.

Let's consider the 2N3904 transistor, where the above expression for $I_C(V_{in})$ holds as long as $0.2\text{V} < V_{out} < 40\text{V}$, and $0.1\text{mA} < I_C < 10\text{mA}$.

(Note that the 2N3904 is a cheap transistor that people often use in personal projects. You can get them for 3 cents each if you buy in bulk.)

a) **Write a symbolic expression for V_{out} as a function of I_C.**

Solution

$V_{out} = V_{DD} - RI_C$ since we have a voltage drop of $I_C R$ across the resistor and the top voltage is V_{DD}.

b) Now let's linearize I_C in the neighborhood of an input voltage V_{in}^* and a specific I_C^*. Assume that you have a found a particular pair of input voltage V_{in}^* and current I_C^* that satisfy the current equation (3).

We can look at nearby input voltages and see how much the current changes. We can write the linearized expression for the collector current around this point as:

$$I_C(V_{in}) = I_C(V_{in}^*) + \delta I_C \approx I_C^* + m(V_{in} - V_{in}^*) = I_C^* + m\ \delta V_{in} \tag{4}$$

where $\delta V_{in} = V_{in} - V_{in}^*$ is the change in input voltage and $\delta I_C = I_C - I_C^*$ is the change in collector current.

What is m here as a function of I_C^* and V_{TH}?

(If you take EE105, you will learn that this m is called the transconductance, which is usually written g_m, and is the single most important parameter in most analog circuit designs.)

(HINT: First just find m by taking the appropriate derivative and using the chain rule as needed. Then leverage the special properties of the exponential function to express it in terms of the desired quantities.)

Solution

We start out by writing out the linearization form that we are looking for.

$$I_C(V_{in}) = I_C^* + \delta I_C = I_C(V_{in}^*) + m\ \delta V_{in}$$

Here, we can isolate the δI_C term by subtracting $I_C^* = I_C(V_{in}^*)$ from both sides.

$$\delta I_C = m\ \delta V_{in}$$

Now, the meaning of the m is the slope of the I_C curve at V_{in}^*.

$$m = dI_C/dV_{in}|_{V_{in}^*} \tag{5}$$

$$= \frac{1}{V_{TH}} I_S e^{\frac{V_{in}}{V_{TH}}}|_{V_{in}^*} \tag{6}$$

$$= \frac{I_C^*}{V_{TH}} \tag{7}$$

where in the last line, we recognize that the expression in the exponential with the I_S before it is just I_C itself. This is why the I_S constant didn't need to be told to you.

We can use these equations to linearize I_c at certain chosen values of V_{in}, such as values $V_{in}^* = 0.65$ V and $V_{in}^* = 0.65$ V given in parts (d) and (e) below. We plot these linearizations here to help visualize our results.

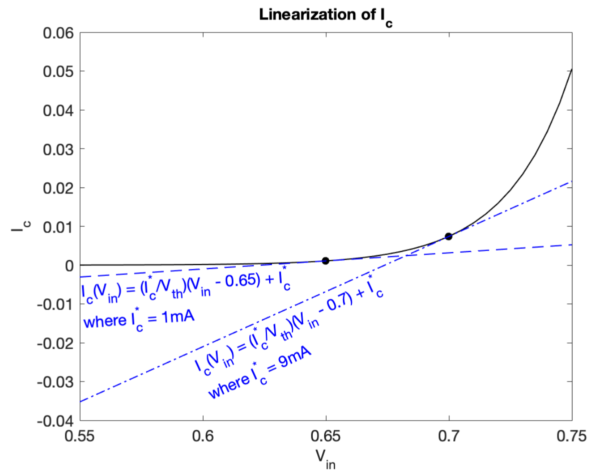


Figure 2: Linearization of I_c

c) We now have a linear relationship between small changes in current and voltage, $\delta I_C = m\ \delta V_{in}$ around a known solution (I_C^*, V_{in}^*). This is called a "bias point" in circuits terminology.

Going back to your equation from part (a), plug in your linearized equation for I_C. Define the appropriate V_{out}^* so that it makes sense to view $V_{out} = V_{out}^* + \delta V_{out}$ when we have $V_{in} = V_{in}^* + \delta V_{in}$, and **find the approximate linear relationship between δV_{out} and δV_{in}.**

The ratio $\frac{\delta V_{out}}{\delta V_{in}}$ is called the small-signal voltage gain of this amplifier around this bias point.

Solution

Expanding out and remembering the equation for V_{out} from above:

$$V_{out} = V_{out}^* + \delta V_{out} = V_{DD} - R(I_C^* + m\ \delta V_{in})$$

Therefore, we define $V_{out}^* = V_{DD} - RI_C^*$ and then

$$\delta V_{out} = -R\ m\ \delta V_{in}$$

with m as above. Namely

$$\delta V_{out} = -\frac{I_C^* R}{V_{TH}}\ \delta V_{in} = -\frac{V_{DD} - V_{out}^*}{V_{TH}}\ \delta V_{in}$$

.

You don't have to simplify it to this point, but this form is useful because it shows you that the gap between the operating point $V_{out}*$ to the supply rail V_{DD} matters to understand the small-signal gain. We want as much current as possible to make the gain big, but there is a limit to how big the current can get.

We can use these equations to linearize V_{out} at certain chosen values of V_{in}, such as values $V_{in}^* = 0.65$ V and $V_{in}^* = 0.65$ V given in parts (d) and (e) below. We plot these linearizations here to help visualize our results. The slope of these lines are the small signal voltage gain $\frac{\delta V_{out}}{\delta V_{in}} = -\frac{I_C^* R}{V_{th}}$.

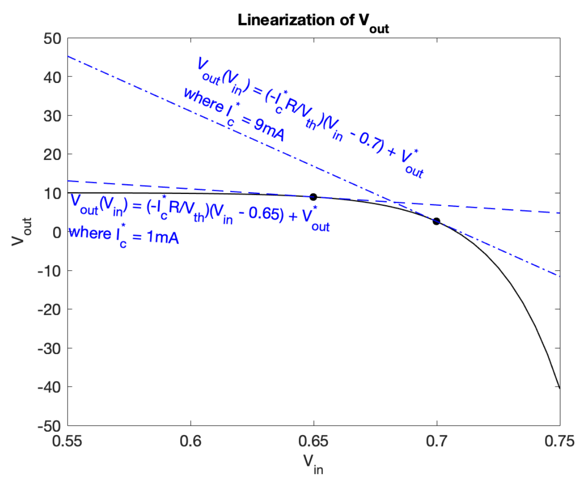


Figure 3: Linearization of V_{out}

d) Assuming that $V_{DD} = 10V$, $R = 1k\Omega$, and $I_C^* = 1mA$ when $V_{in}^* = 0.65V$, **what is the small-signal voltage gain $\frac{\delta V_{out}}{\delta V_{in}}$, between the input and the output around this bias point?** (one or two digits of precision is plenty)

Solution

Just plugging in using the current form:

$$-(1k\Omega)(1mA/26mV) = -1V/26mV \approx -38$$

e) If $I_C^* = 9mA$ when $V_{in}^* = 0.7V$, **what is the small-signal voltage gain around this bias point?** (one or two digits is plenty)

Solution

$$-(1k\Omega)(9mA/26mV) = -9V/26mV \approx -350$$

Notice here that we have V_{out}^* is around $1V$ (compared to $V_{DD} = 10$ V). This is close to as big as this gain can get. It is not obvious that it is actually

V_{DD} and V_{TH} that provide the fundamental limit on the small-signal gain for such circuits, but the simple linearization analysis above reveals this.

4 (OPTIONAL) Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very effective way to really learn material. Having some practice at trying to create problems helps you study for exams much better than simply solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really consolidate your understanding of the course material.

5 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?** List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)

d) **Do you have any feedback on this homework assignment?**

e) **Roughly how many total hours did you work on this homework?**