This homework is designed as a practice for midterm 1.Solutions are published with the homework. Problems 1-3 are designed for midterm practice and problem 4-5 are related to filter and transfer function, materials covered in the class most recently.

There is no self-grade and grading deadline.

1 Matrix Differential Equations

In this problem, we consider ordinary differential equations which can be written in the following form

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \tag{1}$$

where x, y are variables depending on t, $x' = \frac{dx}{dt}$, $y' = \frac{dy}{dt}$, and A is a 2×2 matrix with constant coefficients. We call (1) a matrix differential equation.

a) Suppose we have a system of ordinary differential equations

$$x' = 8x + 7y \tag{2}$$
$$y' = -4x - 3y \tag{3}$$

Write this in the form of (1).

Solution

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ -4 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}, \tag{4}$$

b) Compute the eigenvalues of the matrix A from the previous part.

Solution

The characteristic polynomial of A is

$$\begin{aligned} \det\left(\begin{bmatrix} 8-\lambda & 7 \\ -4 & -3-\lambda \end{bmatrix}\right) &= (8-\lambda)(-3-\lambda) + 28 \\ &= \lambda^2 - 8\lambda + 3\lambda - 24 + 28 \\ &= \lambda^2 - 5\lambda + 4 \\ &= (\lambda - 1)(\lambda - 4). \end{aligned}$$

Thus the eigenvalues of A are $\lambda = 1, 4$.

c) We claim that the solution for $x(t), y(t)$ is of the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_0 t} + c_3 e^{\lambda_1 t} \end{bmatrix},$$

where c_0, c_1, c_2, c_3 are constants, and λ_0, λ_1 are the eigenvalues of A. Suppose that the initial conditions are $x(0) = 1, y(0) = 1$. Solve for the constants c_0, c_1, c_2, c_3.

Solution

Substituting the eigenvalues computed in the previous part, we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_0 e^{t} + c_1 e^{4t} \\ c_2 e^{t} + c_3 e^{4t} \end{bmatrix}.$$

At $t = 0$, we have $x(0) = 1, y(0) = 1, \frac{dx}{dt}(0) = 15, \frac{dy}{dt}(0) = -7$:

$$1 = c_0 + c_1 \tag{5}$$
$$15 = c_0 + 4c_1 \tag{6}$$
$$1 = c_2 + c_3 \tag{7}$$
$$-7 = c_2 + 4c_3 \tag{8}$$

This gives $c_0 = -\frac{11}{3}$, $c_1 = \frac{14}{3}$, $c_2 = \frac{11}{3}$, and $c_3 = -\frac{8}{3}$. Thus we have

$$x(t) = -\frac{11}{3}e^{t} + \frac{14}{3}e^{4t} \tag{9}$$
$$y(t) = \frac{11}{3}e^{t} - \frac{8}{3}e^{4t} \tag{10}$$

d) Verify that the solution for $x(t), y(t)$ found in the previous part satisfies the original system of differential equations (2), (3).

Solution

We compute the derivative of x with respect to t in (9) to get

$$x'(t) = -\frac{11}{3}e^{t} + \frac{56}{3}e^{4t}.$$

The right hand side of (2) is

$$8x + 7y = -\frac{88}{3}e^{t} + \frac{112}{3}e^{4t} + \frac{77}{3}e^{t} - \frac{56}{3}e^{4t} = -\frac{11}{3}e^{t} + \frac{56}{3}e^{4t}$$

hence our solution for $x(t)$ satisfies (2).

Similarly, we compute the derivative of y with respect to t in (10) to get

$$y'(t) = \frac{11}{3}e^{t} - \frac{32}{3}e^{4t}.$$

The right hand side of (3) is

$$-4x - 3y = \frac{44}{3}e^{t} - \frac{56}{3}e^{4t} - \frac{33}{3}e^{t} + \frac{24}{3}e^{4t} = \frac{11}{3}e^{t} - \frac{32}{3}e^{4t}$$

hence our solution for $y(t)$ satisfies (3).

e) We now apply the method above to solve another second-order ordinary differential equation. Suppose we have the system

$$z''(t) - 5z'(t) + 6z(t) = 0, \tag{11}$$

where $z' = \frac{dz}{dt}$ and $z'' = \frac{d^2z}{dt^2}$. Write this in the form of (1), by choosing your state variables to be $x(t) = z(t)$, $y(t) = z'(t)$.

Solution

If we set $x(t) = z(t)$, $y(t) = z'(t)$, then we have

$$x'(t) = z'(t) = y(t) \tag{12}$$

$$y'(t) = z''(t) = 5z'(t) - 6z(t) = 5y(t) - 6x(t) \tag{13}$$

We can write this in the form of (1) as follows

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}, \tag{14}$$

f) Solve the system in (11) with the initial conditions $z(0) = 1$, $z'(0) = 1$, using the method developed in parts (b) and (c).

Solution

We first compute the eigenvalues of the matrix from the previous part. The characteristic polynomial is

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -6 & 5-\lambda \end{bmatrix}\right) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

Thus the eigenvalues are $\lambda = 2, 3$.

From part (c), the solution for $x(t)$, $y(t)$ is of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_0 e^{2t} + c_1 e^{3t} \\ c_2 e^{2t} + c_3 e^{3t} \end{bmatrix}.$$

At $t = 0$, we have $x(0) = z(0) = 1$ and $x'(0) = z'(0) = 1$:

$$1 = c_0 + c_1 \tag{15}$$
$$1 = 2c_0 + 3c_1 \tag{16}$$

This gives $c_0 = 2$ and $c_1 = -1$. Thus we have

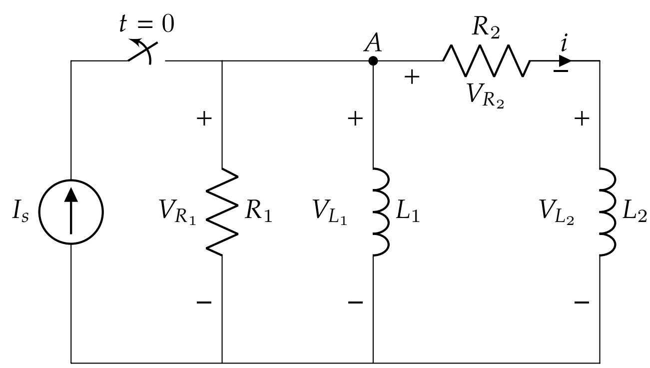
$$x(t) = 2e^{2t} - e^{3t} \tag{17}$$
$$y(t) = x'(t) = 4e^{2t} - 3e^{3t} \tag{18}$$

Hence we have the solution

$$z(t) = 2e^{2t} - e^{3t} .$$

2 RL Circuit

Consider the circuit below



a) What is $i(0)$.

Hint: What is the current flowing through L_1 before the switch opens? Consequently what is the current flowing through L_2?

Solution

At DC, inductors behave like shorts. This means that the current through L_1 is I_s for $t < 0$. Since current across an inductor cannot "jump", $i(0) = 0$.

b) What is $\frac{\mathrm{d}i}{\mathrm{d}t}(0)$?

Solution

At $t = 0$, we know that $i(0) = 0$. This means that $V_{L_2} = V_{L_1} = V_{R_1}$. Additionally, we know that right after the switch flips, all of the current going through L_1 must now pass through R_1. Thus,

$$V_{L_2} = V_{R_1} = -I_s * R_1$$

The equation for an inductor tells us that $V_{L_2} = L_2 \frac{di}{dt}$, so we get that

$$\frac{di}{dt}(0) = \frac{-I_s R_1}{L_2}$$

c) What is the relationship between the voltages across L_1 and R_1?

Solution

They are the same.

d) Use KCL on node A and the relationship derived above to arrive at a differential equation of the form

$$\frac{d^2 i}{dt^2}(t) + a_1 \frac{di}{dt}(t) + a_0 i(t) = 0$$

where $i(t)$ is the current going through L_2.

Solution

Let the current going through R_1 be i_0 and the current going through L_1 be i_1. Then,

$$i_0 + i_1 + i = 0 \implies \frac{di_0}{dt} + \frac{di_1}{dt} + \frac{di}{dt} = 0$$

This means that,

$$\frac{di_0}{dt} = -\frac{di}{dt} - \frac{V_{L_1}}{L_1}$$

or,

$$\frac{di_0}{dt} = -\frac{di}{dt} - i\frac{R_2}{L_1} - \frac{L_2}{L_1}\frac{di}{dt}$$

Now taking the derivative with respect to time of

$$V_{R_1} = V_{R_2} + V_{L_2}$$

we get,

$$R_1 \frac{di_0}{dt} = R_2 \frac{di}{dt} + L_2 \frac{d^2 i}{dt^2}$$

Plugging in the expression for $\frac{\mathrm{d}i_0}{\mathrm{d}t}$, we get,

$$a_1 = \frac{R_1 + R_2}{L_2} + \frac{R_1}{L_1}$$

and

$$a_0 = \frac{R_1 R_2}{L_1 L_2}$$

e) Let $R_1 = R_2 = R$ and $L_1 = L_2 = L$. Recall that the above differential equation can be reshaped into the following linear algebra problem:

$$\begin{bmatrix} \frac{\mathrm{d}i}{\mathrm{d}t} \\ \frac{\mathrm{d}^2 i}{\mathrm{d}t^2} \end{bmatrix} = A \begin{bmatrix} i \\ \frac{\mathrm{d}i}{\mathrm{d}t} \end{bmatrix}$$

What is the A matrix and what are its eigenvalues?

Solution

We have,

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}$$

with,

$$a_1 = \frac{3R}{L}$$

and

$$a_0 = \frac{R^2}{L^2}$$

Thus, the eigenvalues are

$$\lambda = \frac{R}{2L}(-3 \pm \sqrt{5})$$

f) Will this circuit exhibit any oscillations?

Solution

No, since the eigenvalues are real.

g) Now, consider the case when the switch is open for time $t < 0$, and the switch closes at $t = 0$. What is V_{R_1}?

Solution

Since current cannot change instantaneously across inductors, $I_{R_1} = I_s$. Thus,

$$V_{R_1} = I_s R_1$$

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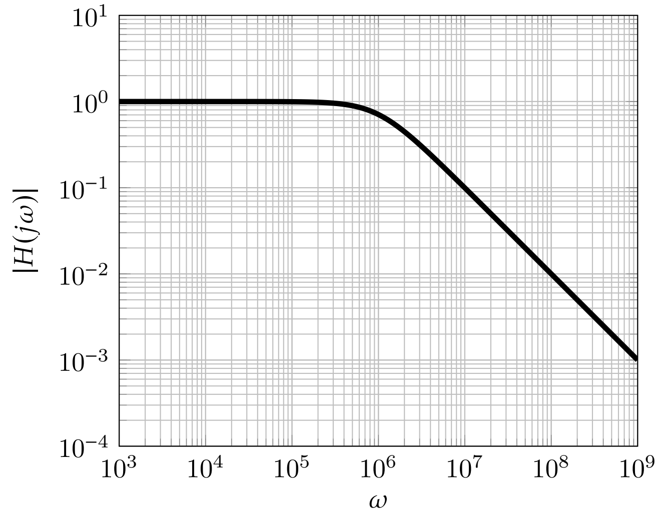
3 Transfer Functions and Filters

a) **Identify each of the Bode Plots, circuits, and transfer functions as either a lowpass or highpass filter.** Indicate your answer by filling in the appropriate bubble.

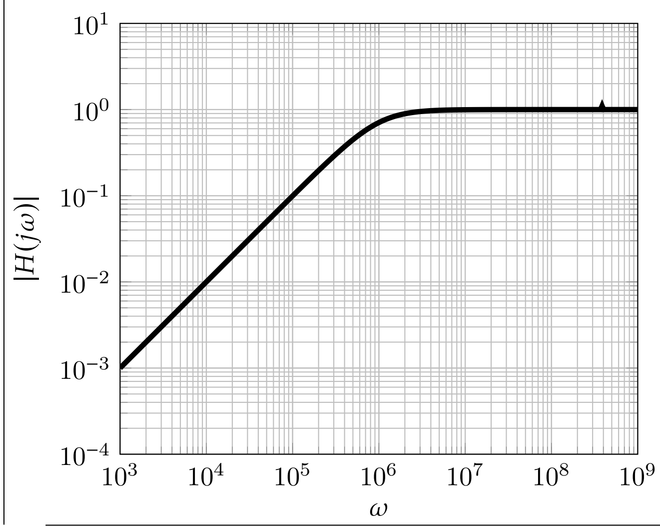
Table 1: Table to be filled in for your answers. Fill in bubbles.

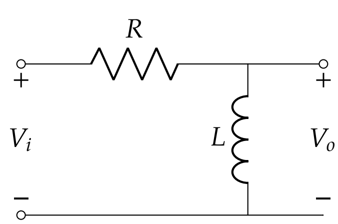
	Lowpass	Highpass
Bode Plot A	◯	◯
Bode Plot B	◯	◯
Circuit C	◯	◯
Circuit D	◯	◯
Transfer Fn E	◯	◯
Transfer Fn F	◯	◯

Bode Plot A ($\omega_c = 10^6$)

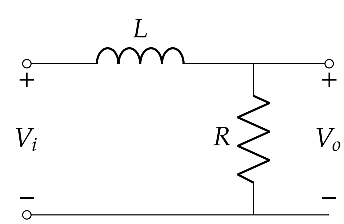


Bode Plot B ($\omega_c = 10^6$)





Circuit C: $H(j\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i}$



Circuit D: $H(j\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i}$

Transfer function E: $H_E(j\omega) = \dfrac{\frac{j\omega}{\omega_c}}{1 + \frac{j\omega}{\omega_c}}$ | Transfer function F: $H_F(j\omega) = \dfrac{1}{1 + \frac{j\omega}{\omega_c}}$

Solution

	Lowpass	Highpass
Graph	A	B
Circuit	D	C
Equation	F	E

b) Consider the three filters in cascade below, with unity-gain op-amp buffers in between them:

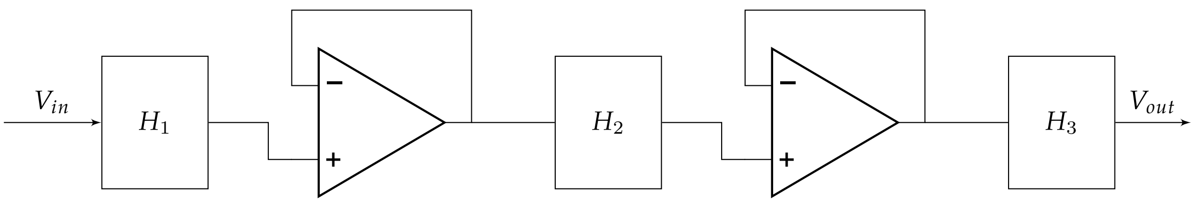


Figure 2: Three filters cascaded via unity-gain op-amp buffers

Suppose that at some frequency ω_0 radians/sec we know that:

$$H_1(j\omega_0) = 3e^{j\frac{\pi}{4}} \quad H_2(j\omega_0) = \frac{1}{2}e^{-j\frac{\pi}{3}} \quad H_3(j\omega_0) = 4e^{j\frac{5\pi}{6}}$$

If $V_{in}(t) = 2\sin\left(\omega_0 t + \frac{\pi}{2}\right)$:

What is the phasor for the input voltage: $\widetilde{V_{in}}$?

What is the phasor for the output voltage: $\widetilde{V_{out}}$?

What is $V_{out}(t)$?

Solution

First we find the phasor representation of $V_{in}(t)$. Remember that $sin(t + \frac{\pi}{2}) = cos(t)$:

$$V_{in}(t) = 2cos(\omega_0 t)$$

$$\widetilde{V_{in}} = 2e^{j0} = 2$$

Then to get the overall transfer function, multply all the individual transfer functions. This is the same as multplying the phasor magnitudes and adding the phases:

$$H_{total} = \frac{V_{out}}{V_{in}} = H_1 \cdot H_2 \cdot H_3 = 6e^{j\frac{3\pi}{4}}$$

From phasor analysis, $\widetilde{V_{out}} = H_{total}\widetilde{V_{in}} = 12e^{j\frac{3\pi}{4}}$

Finally we convert back to the original time domain:

$$V_{out}(t) = 12cos(\omega_0 t + \frac{3\pi}{4})$$

The phase shift $\frac{3\pi}{4}$ is the phase of the output voltage phasor. We know this from the equation for cosine: $\cos\theta = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$.

4 Bandpass Filter: Lowpass and Highpass Cascade

In lecture, you heard about how you can go through the design of a bandpass filter by cascading lowpass and highpass filters via buffers (Op-Amps in unity-gain negative feedback to prevent loading effects). In this problem, you will do this for yourself.

Consider an input signal that is composed of the superposition of:

- 20mV level pure tone at 60Hz corresponding to power line noise.
- 1mV level pure tone at 600Hz corresponding to a voice signal.
- 10mV level pure tone at 60kHz corresponding to fluorescent light noise.

This is the signal that you want to filter.

a) We would like to keep the 600Hz tone, which could correspond to a voice signal, for example.

Ignoring any phase offset for each signal (i.e. set the phases to zero), **write the $V_{in}(t)$ that describes the above input in time domain**.

Solution

Each pure tone in the input signal corresponds to a cosine wave. All of these tones can be added up:

$$V_{in}(t) = (20*10^{-3})\cos(2\pi \cdot 60t)+(1*10^{-3})\cos(2\pi \cdot 600t)+(10*10^{-3})\cos(2\pi \cdot 60000t)$$

b) **What are the radian/sec frequencies ωs involved and the phasors associated with each tone?**

Solution

Remember that the frequencies of the tones were provided in Hz. To convert this to radians/sec use $\omega = 2\pi f$

Power line: 376.99 rad/s

Voice: 3769.9 rad/s

Fluorescent light noise: 376991.1 rad/s

Since the phase of each of these tones is 0, the phasors for each tone are simply the amplitude of the tone multiplied by e^{j0}:

Power line: $20 * 10^{-3} e^{j0} = 20 * 10^{-3}\text{V}$

Voice: $1 * 10^{-3} e^{j0} = 1 * 10^{-3}\text{V}$

Flourescent light noise: $10 * 10^{-3}) e^{j0} = 10 * 10^{-3}\text{V}$

c) To achieve your goal of keeping the voice tone but rejecting the noise from the power-lines and fluorescent lights, **at what frequency do you want to have the knee or cutoff-frequency for the lowpass filters**?

Solution

There are a wide range of possible cutoff-frequencies you can choose for the lowpass filter. You can choose any value in between the voice frequency and the fluorescent light frequency and get some filtering (and get credit). Your specific choice depends on the tradeoffs you may want to take. For example, if you choose a cutoff frequency close to the voice frequency, you will suppress the fluorescent light frequency well, but will also start to attenuate the voice frequency a little. If you choose a cutoff frequency close to the fluorescent light frequency, you will minimize attenuation of the voice frequency, but also have very little suppression of the fluorescent light frequency. Choosing this value in a principled way depends on more contextual information.

One reasonable solution is to choose a frequency that is the geometric mean of the voice and fluorescent frequencies. The geometric mean puts the cutoff frequency half way between the voice and fluorescent frequencies on a log scale:

$$f_c = \sqrt{f_{voice} \cdot f_{fluorescent}} = \sqrt{600 \cdot 60000}\text{Hz} = 6000\text{Hz}$$

and,

$$\omega_c = 2\pi f_c = 2\pi 6000 \frac{\text{rad}}{\text{sec}} = 37699.1 \frac{\text{rad}}{\text{sec}}$$

A second reasonable way to select a cutoff frequency is to note that since 600Hz and 60KHz are fairly far apart, we can simply choose a frequency 10x the voice frequency. This will minimize attenuation of the voice tone while still suppressing the fluorescent light tone:

$$f_c = f_{voice} * 10 = 6KHz \qquad \omega_c = 2\pi f_c = 2\pi 6000 \frac{\text{rad}}{\text{sec}} = 37699.1 \frac{\text{rad}}{\text{sec}}$$

Coincidentally, this is the same frequency as the geometric mean approach.

d) To achieve your goal of keeping the voice tone but rejecting the noise from the power-lines and fluorescent lights, **at what frequency do you want to have the knee or cutoff-frequency for the highpass filters**?

Solution

There are a wide range of possible cutoff-frequencies you can choose for the highpass filter. You can choose any value in between the voice frequency and the power line frequency and get some filtering (and get credit). Your specific choice depends on the tradeoffs you may want to take. For example, if you choose a cutoff frequency close to the voice frequency, you will suppress the power line frequency well, but will also start to attenuate the voice frequency a little. If you choose a cutoff frequency close to the power line frequency, you will minimize attenuation of the voice frequency, but also have very little suppression of the power line frequency. Choosing this value in a principled way depends on more contextual information.

Since 60Hz and 600Hz are fairly close together, one reasonable way to select a cutoff frequency is to choose the geometric mean of the two frequencies:

$$f_c = \sqrt{f_{powerline} f_{voice}} = \sqrt{60 \cdot 600}\text{Hz} \approx 189.7\text{Hz}$$

and,

$$\omega_c = 2\pi f_c = 2\pi * 189 \frac{\text{rad}}{\text{sec}} = 1187.5 \frac{\text{rad}}{\text{sec}}$$

e) Suppose that you only had 1µF capacitors to use. **What resistance values would you choose for your highpass and lowpass filters so that they have the desired cutoff frequencies?**

Solution

For the high pass filter, the cutoff frequency is $\omega_c = \frac{1}{RC}$. Solving for R:

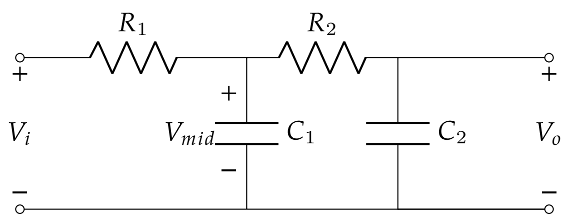
$$R = \frac{1}{C\omega_c}$$

In our case, we selected $\omega_c = 1187.5 rad/s$, so $R_{highpass} = \frac{1}{(1*10^{-6})(1187.5)} = 842\Omega$

For an RC low pass filter, the cutoff frequency expression is the same ($\omega_c = \frac{1}{RC}$), so $R_{lowpass} = \frac{1}{(1*10^{-6})(37699.1)} = 26.5\Omega$

5 Transfer functions and why loading is annoying

Consider the circuit below.



The circuit has an input phasor voltage $\widetilde{V}_i$ at frequency ω rad/sec applied at the input terminals shown in the illustration above, causing an output phasor voltage $\widetilde{V}_o$ at output terminals.

a) We are going to construct the transfer function $H(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i}$ in two steps. We will compute two intermediate transfer functions, $H_1(\omega) = \frac{\widetilde{V}_{mid}}{\widetilde{V}_i}$ and $H_2(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_{mid}}$. Then, we will find the overall transfer function as the product of these two intermediate transfer functions, i.e. $H(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i} = H_1(\omega)H_2(\omega)$.

This approach is valid since the $\widetilde{V}_{mid}$ cancel.

For the first step, **find the intermediate transfer function** $H_2(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_{mid}}$**.** Have your expression be in terms of Z_{R2} and Z_{C2}, that is the impedances of R_2 and C_2.

Solution

V_{mid} and V_o are in a voltage divider configuration, with impedances Z_{R2} and Z_{C2}. This means

$$V_o = (V_{mid})\frac{Z_{C2}}{Z_{R2} + Z_{C2}}, \tag{19}$$

so we can say

$$H_2(\omega) = \frac{V_o}{V_{mid}} = \frac{Z_{C2}}{Z_{R2} + Z_{C2}}. \tag{20}$$

b) Now, **compute the other intermediate transfer function** $H_1(\omega) = \frac{\widetilde{V}_{mid}}{\widetilde{V}_i}$. Have your expression be in terms of Z_{R1}, Z_{R2}, Z_{C1}, and Z_{C2}. (i.e. Don't forget to consider the impact of loading by R_2 and C_2 in this transfer function.) *hint: Applying KCL at the V_{mid} node would be a good place to start. You should try to find an expression for H_1 that has factors that H_2 can cancel out.*

Solution

From KCL, we have

$$\frac{V_{mid} - V_i}{Z_{R1}} + \frac{V_{mid}}{Z_{C1}} + \frac{V_{mid}}{Z_{R2} + Z_{C2}} = 0 \tag{21}$$

$$V_{mid}\left(\frac{1}{Z_{R1}} + \frac{1}{Z_{C1}} + \frac{1}{Z_{R2} + Z_{C2}}\right) = V_i\left(\frac{1}{Z_{R1}}\right) \tag{22}$$

$$H_1(\omega) = \frac{V_{mid}}{V_i} = \frac{1}{Z_{R1}\left(\frac{1}{Z_{R1}} + \frac{1}{Z_{C1}} + \frac{1}{Z_{R2}+Z_{C2}}\right)} \tag{23}$$

c) Then, **use these two intermediate transfer functions to calculate the overall transfer function** as $H(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i} = H_1(\omega)H_2(\omega)$.

Solution

Combining the above formulas, we have

$$\begin{aligned}
H(\omega) &= \frac{Z_{C2}}{Z_{R2} + Z_{C2}} \frac{1}{Z_{R1}\left(\frac{1}{Z_{R1}} + \frac{1}{Z_{C1}} + \frac{1}{Z_{R2}+Z_{C2}}\right)} && (24) \\
&= \frac{Z_{C2}}{Z_{R2} + Z_{C2}} \frac{1}{1 + \frac{Z_{R1}}{Z_{C1}} + \frac{Z_{R1}}{Z_{R2}+Z_{C2}}} && (25) \\
&= \frac{Z_{C2}}{Z_{R2} + Z_{C2} + \frac{Z_{R2}+Z_{C2}}{Z_{C1}} Z_{R_1} + Z_{R1}} && (26) \\
&= \frac{Z_{C1}Z_{C2}}{Z_{R1}Z_{C1} + Z_{C1}Z_{C2} + Z_{R1}Z_{C2} + Z_{R2}Z_{C1} + Z_{R1}Z_{R2}}. && (27)
\end{aligned}$$

d) Sometimes it is useful to collect all the frequency dependence into one place and to figure out how to think about what scale might be somewhat natural for the frequency. **Obtain an expression for $H(\omega) = \widetilde{V}_o/\widetilde{V}_i$ in the form**

$$H(\omega) = \frac{V_o}{V_i} = \frac{1}{1 + 2\xi\frac{j\omega}{\omega_c} + \frac{(j\omega)^2}{\omega_c^2}},$$

given that $R_1 = 2\,\Omega$, $R_2 = 4\,\Omega$, $C_1 = \frac{9}{2}\,\mathrm{F}$, and $C_2 = 1\,\mathrm{F}$. What are the values of ξ and ω_c?

Solution

The impedance in the phasor domain is given by

$$Z_{R1} = R_1 \qquad Z_{R2} = R_2 \qquad Z_{C1} = \frac{1}{j\omega C_1} \qquad Z_{C2} = \frac{1}{j\omega C_2}$$

Using the result in part (a), we have

$$H(\omega) = \frac{\frac{1}{j\omega C_1}\frac{1}{j\omega C_2}}{\frac{R_1}{j\omega C_1} + \frac{1}{j\omega C_1}\frac{1}{j\omega C_2} + \frac{R_1}{j\omega C_2} + \frac{R_2}{j\omega C_1} + R_1 R_2} \tag{28}$$

$$= \frac{1}{1 + j\omega\big(R_1C_1 + (R_1 + R_2)C_2\big) + (j\omega)^2 R_1R_2C_1C_2} \tag{29}$$

$$= \frac{1}{1 + j\omega(15) + (j\omega)^2(36)} \tag{30}$$

$$= \frac{1}{1 + 2(15/12)\frac{j\omega}{1/6} + (\frac{j\omega}{1/6})^2}, \tag{31}$$

Which means $\xi = \frac{15}{12} = \frac{5}{4}$ and $\omega_c = \frac{1}{6}$.

e) **For the previous case, what is the magnitude of the transfer function at the $\omega = \omega_c$ you calculated?**

This is here so that you can see that just because we called it ω_c doesn't mean that the amplitude here is $\frac{1}{\sqrt{2}}$.

Solution

At $\omega = \omega_c$, the transfer function becomes:

$$H(\omega_c) = \frac{1}{2\xi j}$$

Thus using $\xi = \frac{5}{4}$, the magnitude of the transfer function is

$$|H(\omega_c)| = \frac{1}{2\xi} = \frac{1}{2 \cdot \frac{5}{4}} = \frac{4}{10} = 0.4$$

f) We can express the transfer function $H(\omega)$ in the polar form. That is,

$$H(\omega) = M(\omega)e^{j\phi(\omega)}$$

The functions $M(\omega)$ and $\phi(\omega)$ are the magnitude and the phase angle of $H(\omega)$, respectively. **Write down $M(\omega)$ and $\phi(\omega)$ using the transfer function you derived in part (b).**

Solution

Rewriting the result in part (b), we have

$$H(\omega) = \frac{1}{1 - 36\omega^2 + 15j\omega} \tag{32}$$

Taking the magnitude, we have

$$M(\omega) = \frac{1}{\sqrt{(1 - 36\omega^2)^2 + (15\omega)^2}} \tag{33}$$

We can find the phase of the transfer function as the negative of the phase of the denominator, i.e.

$$\angle H(\omega) = -\angle(1 - 36\omega^2 + 15j\omega) = -\text{atan2}(15\omega, 1 - 36\omega^2). \tag{34}$$

A solution using $\tan^{-1}(\cdot)$ is also acceptable, but you have to be careful: it is only correct for complex numbers in the first or the fourth quadrants of the complex plane. For complex numbers in the second and the third quadrants, we need to shift their $\tan^{-1}(\cdot)$ by π. In other words, we need

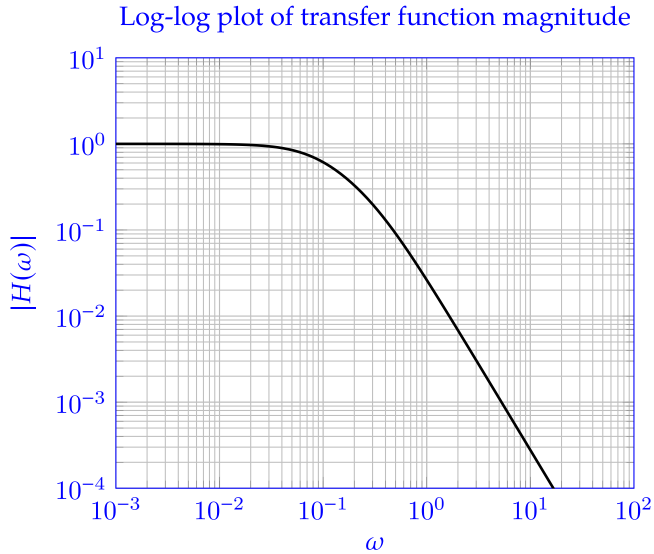
$$\phi(\omega) = \begin{cases} -\tan^{-1}\left(\frac{15\omega}{1-36\omega^2}\right) & \text{when } 0 \leq \omega \leq \frac{1}{6} \\ -\tan^{-1}\left(\frac{15\omega}{1-36\omega^2}\right) - \pi & \text{when } \omega > \frac{1}{6} \end{cases} \tag{35}$$

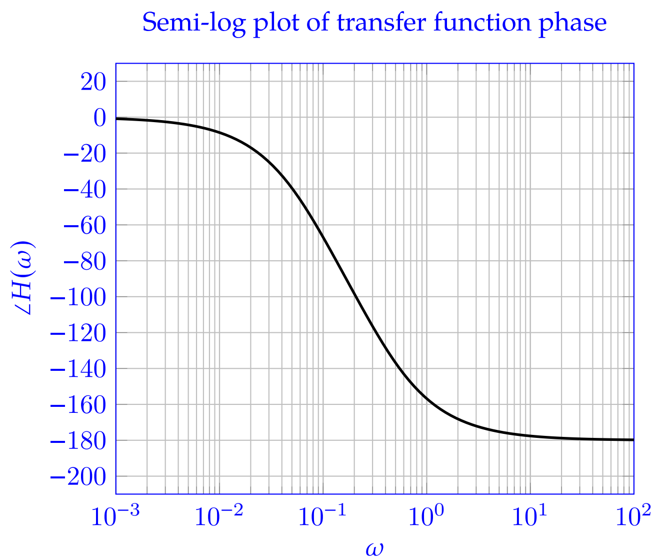
Note that $1 - 36\omega^2 + j15\omega$ is in the first quadrant when $\omega < 1/6$ and in the second quadrant when $\omega > 1/6$.

g) Use a computer and then **draw Bode Plots of $|H(\omega)|$ and $\angle H(\omega)$.**

Solution

Evaluating the $M(\omega)$ and $\phi(\omega)$


Log-log plot of transfer function magnitude
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$|H(\omega)|$
10^{-3} 10^{-2} 10^{-1} 10^0 10^1 10^2
ω


Semi-log plot of transfer function phase
20
0
−20
−40
−60
−80
−100
−120
−140
−160
−180
−200
$\angle H(\omega)$
10^{-3} 10^{-2} 10^{-1} 10^0 10^1 10^2
ω