

This homework is due on Wednesday, February 19, 2020, at 11:59PM.

Self-grades are due on Monday, February 24, 2019, at 11:59PM.

1 Complex Numbers

A common way to visualize complex numbers is to use the complex plane. Recall that a complex number z is often represented in Cartesian form.

$$z = x + jy \text{ with } \operatorname{Re}\{z\} = x \text{ and } \operatorname{Im}\{z\} = y$$

See Figure 1 for a visualization of z in the complex plane.

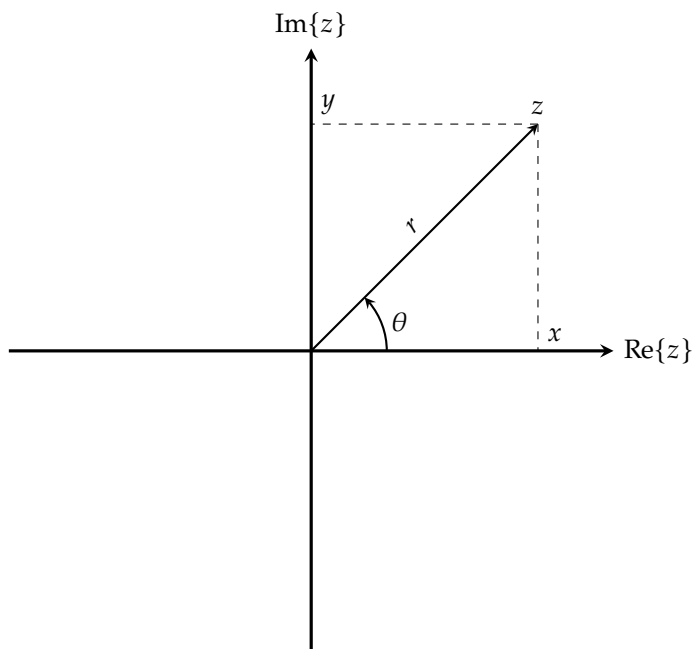


Figure 1: Complex Plane

In this question, we will derive the polar form of a complex number and use this form to make some interesting conclusions.

- a) **Calculate the length of z in terms of x and y as shown in Figure 1.** This is the magnitude of a complex number and is denoted by $|z|$ or r .

(Hint: Use the Pythagorean theorem.)

Solution

$$r = \sqrt{x^2 + y^2} = |z|$$

- b) **Represent x , the real part of z , and y , the imaginary part of z , in terms of r and θ .**

Solution

$$x = r \cos(\theta) \text{ and } y = r \sin(\theta)$$

- c) **Substitute for x and y in z . Use Euler's identity¹ $e^{j\theta} = \cos \theta + j \sin \theta$ to conclude that,**

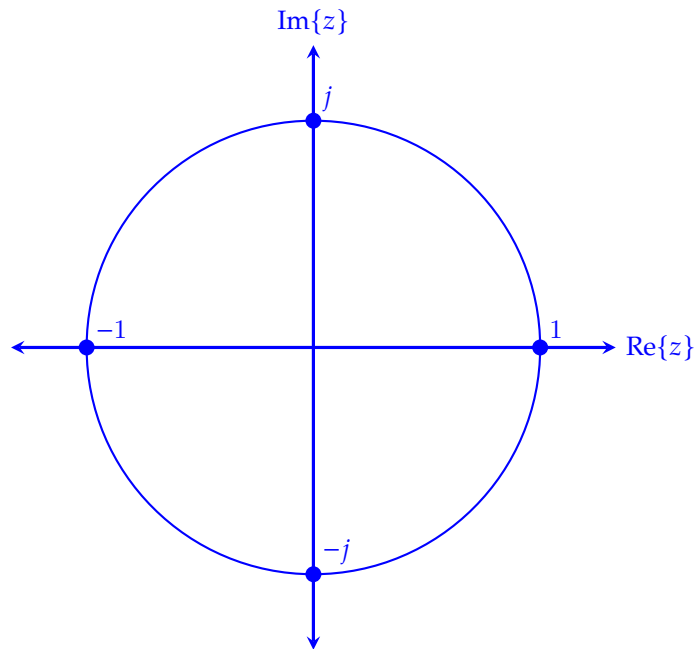
$$z = re^{j\theta}.$$

Solution

$$\begin{aligned} z &= r \cos(\theta) + jr \sin(\theta) \\ &= r(\cos(\theta) + j \sin(\theta)) \\ &= re^{j\theta} \end{aligned}$$

- d) **In the complex plane, sketch the set of all the complex numbers such that $|z| = 1$. What are the z values where the sketched figure intersects the real axis and the imaginary axis?**

¹also known as de Moivre's Theorem.

Solution

- e) If $z = re^{j\theta}$, **prove that** $\bar{z} = re^{-j\theta}$. Recall that the complex conjugate of a complex number $z = x + jy$ is $\bar{z} = x - jy$.

Solution

$$\begin{aligned}\bar{z} &= \overline{r(\cos(\theta) + j \sin(\theta))} \\ &= r(\cos(\theta) - j \sin(\theta)) \\ &= r(\cos(-\theta) + j \sin(-\theta)) \\ &= re^{-j\theta}\end{aligned}$$

- f) **Show (by direct calculation) that,**

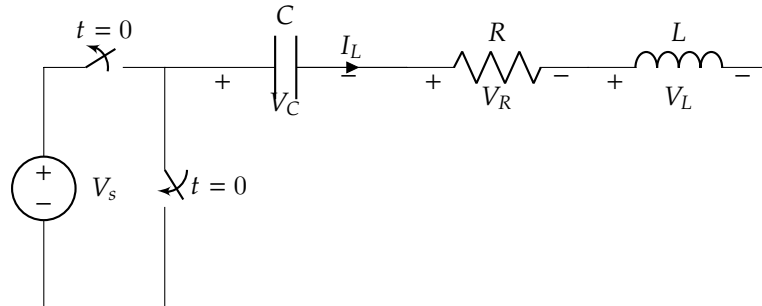
$$r^2 = z\bar{z}.$$

Solution

$$z\bar{z} = re^{j\theta}re^{-j\theta} = r^2e^{j\theta-j\theta} = r^2e^0 = r^2$$

2 RLC Responses: Initial Part

Consider the following circuit like you saw in lecture:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

In this problem, the current through the inductor and the voltage across the capacitor are the natural physical state variables since these are what correlate to how energy is actually stored in the system. (A magnetic field through the inductor and an electric field within the capacitor.)

- a) Write the system of differential equations in terms of state variables $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ that describes this circuit for $t \geq 0$. Leave the system symbolic in terms of V_s , L , R , and C .

Solution

For this part, we need to find two differential equations, each including a derivative of one of the state variables.

First, let's consider the capacitor equation $I_C(t) = C \frac{d}{dt} V_C(t)$. In this circuit, $I_C(t) = I_L(t)$, so we can write

$$I_C(t) = C \frac{d}{dt} V_C(t) = I_L(t) \quad (1)$$

$$\frac{d}{dt} V_C(t) = \frac{1}{C} I_L(t). \quad (2)$$

If we use the state variable names, we can write this as

$$\frac{d}{dt} x_2(t) = \frac{1}{C} x_1(t), \quad (3)$$

so now we have one differential equation.

For the other differential equation, we can apply KVL around the single loop in this circuit. (Alternatively, we could just solve it directly and

substitute in for the desired voltage on the capacitor, which is a state variable.) Going clockwise, we have

$$V_C(t) + V_R(t) + V_L(t) = 0. \quad (4)$$

Using Ohm's Law and the inductor equation $V_L = L \frac{d}{dt} I_L(t)$, we can write this as

$$V_C(t) + R I_L(t) + L \frac{d}{dt} I_L(t) = 0, \quad (5)$$

which we can rewrite as

$$\frac{d}{dt} I_L(t) = -\frac{R}{L} I_L(t) - \frac{1}{L} V_C(t). \quad (6)$$

If we use the state variable names, this becomes

$$\frac{d}{dt} x_1(t) = -\frac{R}{L} x_1(t) - \frac{1}{L} x_2(t), \quad (7)$$

and we have a second differential equation.

To summarize the final system is

$$\frac{d}{dt} x_1(t) = -\frac{R}{L} x_1(t) - \frac{1}{L} x_2(t) \quad (8)$$

$$\frac{d}{dt} x_2(t) = \frac{1}{C} x_1(t). \quad (9)$$

- b) Write the system of equations in vector/matrix form with the vector state variable $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. This should be in the form $\frac{d}{dt} \vec{x}(t) = A \vec{x}(t)$ with a 2×2 matrix A .

Solution

By inspection from the previous part, we have

$$\begin{bmatrix} \frac{d}{dt} x_1(t) \\ \frac{d}{dt} x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad (10)$$

which is in the form $\frac{d}{dt} \vec{x}(t) = A \vec{x}(t)$, with

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}. \quad (11)$$

- c) Find the eigenvalues of the A matrix symbolically.
(Hint: the quadratic formula will be involved.)

Solution

To find the eigenvalues, we'll solve $\det(A - \lambda I) = 0$. In other words, we want to find λ such that

$$\det(A - \lambda I) = \det \left(\begin{bmatrix} -\frac{R}{L} - \lambda & -\frac{1}{L} \\ \frac{1}{C} & -\lambda \end{bmatrix} \right) \quad (12)$$

$$= -\lambda \left(-\frac{R}{L} - \lambda \right) + \frac{1}{LC} \quad (13)$$

$$= \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0. \quad (14)$$

The Quadratic Formula gives

$$\lambda = -\frac{1}{2} \frac{R}{L} \pm \frac{1}{2} \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}. \quad (15)$$

- d) Under what condition on the circuit parameters R, L, C are there going to be a pair of distinct purely real eigenvalues of A ?

Solution

For both eigenvalues to be real and distinct, we need the quantity inside the square root to be positive. In other words, we need

$$\frac{R^2}{L^2} - \frac{4}{LC} > 0, \quad (16)$$

or, equivalently,

$$R > 2\sqrt{\frac{L}{C}}. \quad (17)$$

- e) Under what condition on the circuit parameters R, L, C are there going to be a pair of purely imaginary eigenvalues of A ?

Solution

The only way for both eigenvalues to be purely imaginary is to have $R = 0$. In this case, the eigenvalues would be

$$\lambda = \pm j\sqrt{\frac{1}{LC}}. \quad (18)$$

- f) Assuming that the circuit parameters are such that there are a pair of (potentially complex) eigenvalues λ_1, λ_2 so that $\lambda_1 \neq \lambda_2$, find eigenvectors $\vec{v}_{\lambda_1}, \vec{v}_{\lambda_2}$ corresponding to them.

(HINT: Rather than trying to find the relevant nullspaces, etc., try to find eigenvectors of the form $\begin{bmatrix} 1 \\ y \end{bmatrix}$ where we just want to find the missing entry y . Can you see from the structure of the A matrix why we might want to try that guess?)

Solution

The easy way is just to remember what an eigenvector is. We want $A\vec{v}_{\lambda_i} = \lambda_i\vec{v}_{\lambda_i}$. So, we can try to follow the hint:

$$\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix} = \lambda_i \begin{bmatrix} 1 \\ y \end{bmatrix} = \begin{bmatrix} \lambda_i \\ (\lambda_i)(y) \end{bmatrix} \quad (19)$$

We also know that:

$$\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} - \frac{y}{L} \\ \frac{1}{C} \end{bmatrix} \quad (20)$$

Equating the two equations from above gives:

$$\begin{bmatrix} \lambda_i \\ (\lambda_i)(y) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} - \frac{y}{L} \\ \frac{1}{C} \end{bmatrix} \quad (21)$$

From the second row we see that $y = \frac{1}{\lambda_i C}$. Now we find the eigenvectors as:

$$\vec{v}_{\lambda_1} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_1 C} \end{bmatrix}$$

$$\vec{v}_{\lambda_2} = \begin{bmatrix} 1 \\ \frac{1}{\lambda_2 C} \end{bmatrix}$$

Alternatively, you can try to use the standard approach of finding the nullspace of $A - \lambda_i I$ to arrive at the same answer as above.

- g) Assuming circuit parameters such that the two eigenvalues of A are distinct, let $V = [\vec{v}_{\lambda_1}, \vec{v}_{\lambda_2}]$ be a specific eigenbasis. Consider a coordinate system for which we can write $\vec{x}(t) = V\tilde{\vec{x}}(t)$. What is the \tilde{A} so that $\frac{d}{dt}\tilde{\vec{x}}(t) = \tilde{A}\tilde{\vec{x}}(t)$? It is fine to have your answer expressed symbolically using λ_1, λ_2 .

Solution

V is given by:

$$V = \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix}$$

We know that V transforms from the \tilde{x} coordinate frame to the x coordinate frame, V^{-1} transforms back, and A takes gives the relationship from x to $\frac{d}{dt}x$.

Therefore to go from \tilde{x} to $\frac{d}{dt}\tilde{x}$:

$$\tilde{A} = V^{-1}AV = \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix}$$

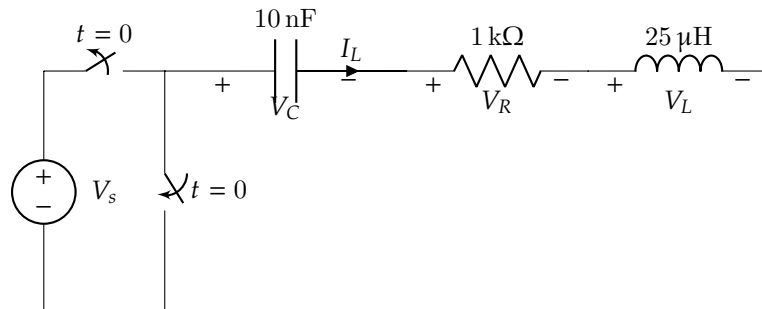
$$\tilde{A} = V^{-1}AV = \frac{\lambda_1 \lambda_2 C}{\lambda_1 - \lambda_2} \begin{bmatrix} \frac{1}{\lambda_2 C} & -1 \\ -\frac{1}{\lambda_1 C} & 1 \end{bmatrix} \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1}{\lambda_1 C} & \frac{1}{\lambda_2 C} \end{bmatrix}$$

$$\tilde{A} = V^{-1}AV = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

You didn't have to multiply things out explicitly. You could have just noticed that the A matrix times the V matrix would give columns that were λ_i times \vec{v}_i each, and then multiplying that by V^{-1} would just pick out the λ_i on the diagonals and zeros on the off-diagonals since $V^{-1}V = I$.

3 RLC Responses: Overdamped Case

Building on the previous problem, consider the following circuit with specified component values:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2.

- a) Suppose $R = 1 \text{ k}\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1 \text{ Volt}$. Find the initial conditions for $\vec{x}(0)$. Recall that \vec{x} is in the changed “nice” eigenbasis coordinates from the first problem.

Solution

First of all, we must state the initial conditions for $\vec{x}(0)$. If the circuit is in steady state before $t = 0$, then no current is flowing and the entire voltage drop is across the capacitor. Therefore:

$$\begin{aligned}x_1(0) &= I_L(0) = 0 \\x_2(0) &= V_C(0) = V_s = 1\end{aligned}$$

Under these conditions, we can solve for

$$\lambda_1 = -1.0 \times 10^5, \lambda_2 = -4.0 \times 10^7$$

$$V^{-1} = \begin{bmatrix} -0.0025 & -0.001 \\ 1.0025 & 0.001 \end{bmatrix}$$

$$\vec{x}(0) = V^{-1} \vec{x}(0) = \begin{bmatrix} -0.0025 & -0.001 \\ 1.0025 & 0.001 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.001 \\ 0.001 \end{bmatrix}$$

- b) Continuing the previous part, find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$.

Solution

Plugging in for the component values gives:

$$\tilde{A} = \begin{bmatrix} -1.0 \times 10^5 & 0 \\ 0 & -4.0 \times 10^7 \end{bmatrix}$$

These eigenvalues are the negative reciprocals of the relevant time constants for these modes.

$$\begin{bmatrix} \frac{d}{dt} \tilde{x}_1(t) \\ \frac{d}{dt} \tilde{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1.0 \times 10^5 & 0 \\ 0 & -4.0 \times 10^7 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix}, \quad (22)$$

Therefore:

$$\tilde{x}_1(t) = K_1 e^{-1.0 \times 10^5 t}$$

$$\tilde{x}_2(t) = K_2 e^{-4.0 \times 10^7 t}$$

Solving for K with the initial condition gives:

$$\tilde{x}_1(t) = -0.001 e^{-1.0 \times 10^5 t}$$

$$\tilde{x}_2(t) = 0.001 e^{-4.0 \times 10^7 t}$$

Converting back to the (\vec{x}) coordinates:

$$\vec{x}(t) = V \vec{\tilde{x}}(t) = \begin{bmatrix} 1 & 1 \\ -1000 & -2.5 \end{bmatrix} \vec{\tilde{x}}(t)$$

$$x_1(t) = -0.001 e^{-1.0 \times 10^5 t} + 0.001 e^{-4.0 \times 10^7 t}$$

$$x_2(t) = e^{-1.0 \times 10^5 t} - 0.0025 e^{-4.0 \times 10^7 t}$$

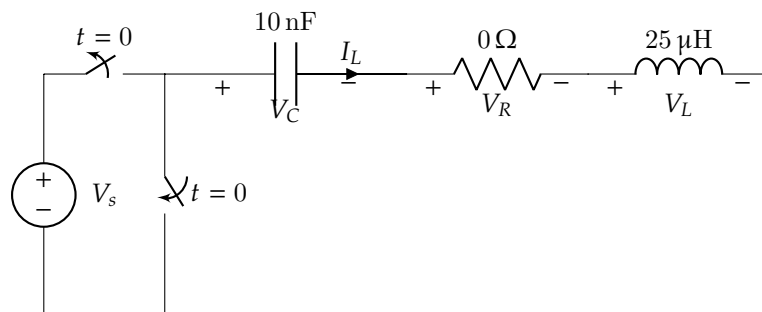
- c) In the provided Jupyter notebook, move the sliders to approximately $R = 1k\Omega$ and $C = 10nF$. Sketch $V_c(t)$ and comment on its appearance. Additionally, sketch the location of the eigenvalues on the complex plane.

Solution

$V_c(t)$ should look like a decaying exponential. The eigenvalues lie on the real axis at coordinates $(-1 \times 10^5, 0)$ and $(-4 \times 10^7, 0)$.

4 RLC Responses: Undamped Case

Building on the previous problem, consider the following circuit with specified component values:



Assume that the capacitor is charged to V_s and there is no current in the inductor for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2.

- a) Suppose $R = 0 \text{ k}\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1 \text{ Volt}$. Find the initial conditions for $\vec{\tilde{x}}(0)$. Recall that $\vec{\tilde{x}}$ is in the changed “nice” eigenbasis coordinates from the first problem.

Solution

Under these conditions, we can solve for $\lambda = \pm j\sqrt{\frac{1}{LC}} = \pm j\sqrt{\frac{1}{250 \times 10^{-15}}} = \pm j2 \times 10^6$

$$\lambda_1 = j2 \times 10^6, \lambda_2 = -j2 \times 10^6$$

Using the rule we derived earlier for finding V , we have

$$V = \begin{bmatrix} 1 & 1 \\ -50j & 50j \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} 0.5 & 0.01j \\ 0.5 & -0.01j \end{bmatrix}$$

which lets us say

$$\vec{\tilde{x}}(0) = V^{-1}\vec{x}(0) = \begin{bmatrix} 0.5 & 0.01j \\ 0.5 & -0.01j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.01j \\ -0.01j \end{bmatrix}$$

- b) Continuing the previous part, find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$.

Solution

Plugging in for the component values gives:

$$\tilde{A} = \begin{bmatrix} j2 \times 10^6 & 0 \\ 0 & -j2 \times 10^6 \end{bmatrix}$$

$$\begin{bmatrix} \frac{d}{dt}\tilde{x}_1(t) \\ \frac{d}{dt}\tilde{x}_2(t) \end{bmatrix} = \begin{bmatrix} j2 \times 10^6 & 0 \\ 0 & -j2 \times 10^6 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix}, \quad (23)$$

Therefore:

$$\tilde{x}_1(t) = K_1 e^{+j2 \times 10^6 t}$$

$$\tilde{x}_2(t) = K_2 e^{-j2 \times 10^6 t}$$

Solving for K with the initial condition gives:

$$\tilde{x}_1(t) = 0.01 j e^{+j2 \times 10^6 t}$$

$$\tilde{x}_2(t) = -0.01 j e^{-j2 \times 10^6 t}$$

Converting back to the (\vec{x}) coordinates:

$$\vec{x}(t) = V \vec{\tilde{x}}(t) = \begin{bmatrix} 1 & 1 \\ -50j & 5 \times 50j \end{bmatrix} \begin{bmatrix} 0.01 j e^{+j2 \times 10^6 t} \\ -0.01 j e^{-j2 \times 10^6 t} \end{bmatrix}$$

$$x_1(t) = 0.01 j e^{+j2 \times 10^6 t} - 0.01 j e^{-j2 \times 10^6 t} = -0.02 \sin(2 \times 10^6 t)$$

$$x_2(t) = 0.5 e^{+j2 \times 10^6 t} + 0.5 e^{-j2 \times 10^6 t} = \cos(2 \times 10^6 t)$$

- c) In the provided Jupyter notebook, move the sliders to approximately $R = 0\Omega$ and $C = 10nF$. Sketch $V_c(t)$ and comment on its appearance. Additionally, sketch the location of the eigenvalues on the complex plane. Are the waveforms for $x_1(t)$ and $x_2(t)$ “transient” — do they die out with time?

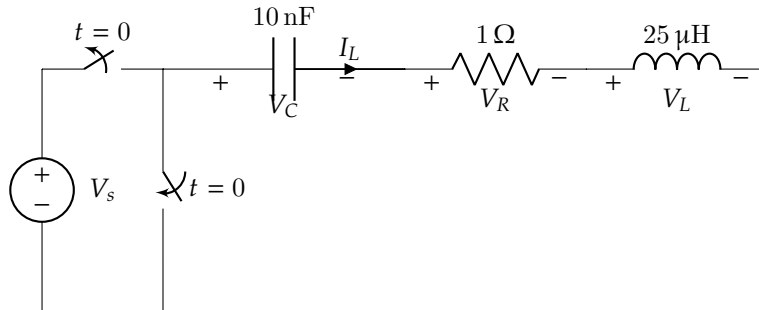
Note: Because there is no resistance, this is called the “undamped” case.

Solution

No, these waveforms are sinusoids and do not die out over time. They are not transient. The eigenvalues are located on the imaginary axis at coordinates $(0, -2 \times 10^6)$ and $(0, 2 \times 10^6)$.

5 RLC Responses: Underdamped Case

Building on the previous problem, consider the following circuit with specified component values:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may round numbers to make the algebra more simple.

- a) Now suppose that $R = 1 \Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1$ Volt. Find the initial conditions for $\tilde{\mathbf{x}}(0)$. Recall that $\tilde{\mathbf{x}}$ is in the changed “nice” eigenbasis coordinates from the first problem.

Solution

Under these conditions, we can solve for

$$\lambda_1 = -0.02 \times 10^6 + j2 \times 10^6, \quad \lambda_2 = -0.02 \times 10^6 - j2 \times 10^6$$

$$V = \begin{bmatrix} 1 & 1 \\ \frac{1}{-0.0002 + j0.02} & \frac{1}{-0.0002 - j0.02} \end{bmatrix}$$

$$\tilde{\mathbf{x}}(0) = V^{-1} \mathbf{x}(0) = \begin{bmatrix} j0.01 \\ -j0.01 \end{bmatrix}$$

- b) Continuing the previous part, find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$.

(HINT: Remember that $e^{a+jb} = e^a e^{jb}$.)

Solution

$$\tilde{A} = \begin{bmatrix} -0.02 \times 10^6 + j2 \times 10^6 & 0 \\ 0 & -0.02 \times 10^6 - j2 \times 10^6 \end{bmatrix}$$

$$\begin{bmatrix} \frac{d}{dt} \tilde{x}_1(t) \\ \frac{d}{dt} \tilde{x}_2(t) \end{bmatrix} = \begin{bmatrix} -0.02 \times 10^6 + j2 \times 10^6 & 0 \\ 0 & -0.02 \times 10^6 - j2 \times 10^6 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix}, \quad (24)$$

Therefore:

$$\begin{aligned} \tilde{x}_1(t) &= K_1 e^{(-0.02 \times 10^6 + j2 \times 10^6)t} \\ \tilde{x}_2(t) &= K_2 e^{(-0.02 \times 10^6 - j2 \times 10^6)t} \end{aligned}$$

Solving for K with the initial condition gives:

$$\begin{aligned} \tilde{x}_1(t) &= j0.01 e^{(-0.02 \times 10^6 + j2 \times 10^6)t} \\ \tilde{x}_2(t) &= -j0.01 e^{(-0.02 \times 10^6 - j2 \times 10^6)t} \end{aligned}$$

Converting back to the (\vec{x}) coordinates:

$$\vec{x}(t) = V \vec{\tilde{x}}(t) = \begin{bmatrix} 1 & 1 \\ -0.5 - j50 & -0.5 + j50 \end{bmatrix} \vec{\tilde{x}}(t)$$

$$x_1(t) = j0.01 e^{(-0.02 \times 10^6 + j2 \times 10^6)t} - j0.01 e^{(-0.02 \times 10^6 - j2 \times 10^6)t}$$

$$x_2(t) = (0.5 - j0.005) e^{(-0.02 \times 10^6 + j2 \times 10^6)t} + (0.5 + j0.005) e^{(-0.02 \times 10^6 - j2 \times 10^6)t}$$

$$x_1(t) = j0.01 e^{-0.02 \times 10^6 t} e^{j2 \times 10^6 t} - j0.01 e^{-0.02 \times 10^6 t} e^{-j2 \times 10^6 t}$$

$$x_2(t) = (0.5 - j0.005) e^{-0.02 \times 10^6 t} e^{j2 \times 10^6 t} + (0.5 + j0.005) e^{-0.02 \times 10^6 t} e^{-j2 \times 10^6 t}$$

$$x_1(t) = e^{-0.02 \times 10^6 t} \left(j0.01 e^{j2 \times 10^6 t} - j0.01 e^{-j2 \times 10^6 t} \right) = -0.020002 e^{-0.02 \times 10^6 t} \sin(2 \times 10^6 t)$$

$$\begin{aligned} x_2(t) &= e^{-0.02 \times 10^6 t} \left((0.5 - j0.005) e^{j2 \times 10^6 t} + (0.5 + j0.005) e^{-j2 \times 10^6 t} \right) \\ &= e^{-0.02 \times 10^6 t} \cos(2 \times 10^6 t) + 0.01 \cdot e^{-0.02 \times 10^6 t} \sin(2 \times 10^6 t). \end{aligned}$$

- c) In the provided Jupyter notebook, move the sliders to approximately $R = 1\Omega$ and $C = 10nF$. Sketch $V_c(t)$ and comment on its appearance. Additionally, sketch the location of the eigenvalues on the complex plane. Are the waveforms for $x_1(t)$ and $x_2(t)$ “transient” — do they die out with time?

Note: Because the resistance is so small, this is called the “underdamped” case. It is good to reflect upon these waveforms to see why engineers consider such behavior to be reflective of systems that don’t have enough damping.

Solution

Yes, the waveforms are transient. They appear to be sinusoids that are decaying exponentially. The eigenvalues should be located at coordinates $(-0.02 \times 10^6, 2 \times 10^6)$ and $(-0.02 \times 10^6, -2 \times 10^6)$.

- d) Notice that you got answers in terms of complex exponentials. Why did the final voltage and current waveforms end up being purely real?

Solution

In this case, it's because of the complex conjugacy of the quantities in the problem. The eigenvalues and their associated eigenvectors were complex conjugates, as were the transformed solutions $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$. When we applied the inverse transformation to $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$, we added together many complex conjugate terms, and the imaginary parts cancelled out.

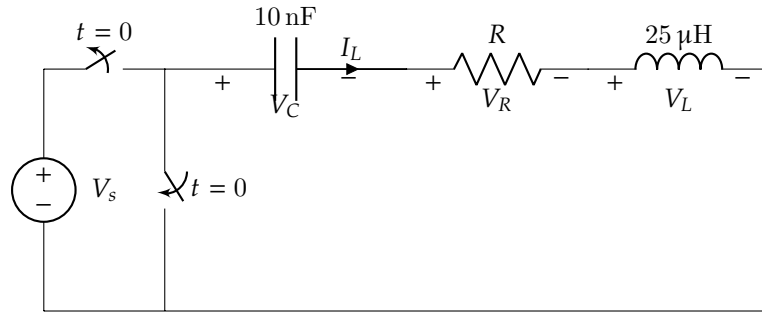
Now, was this just a fluke that just happened to line up perfectly? Is there some A matrix out there with real-valued entries that will result in a complex solution? Or is something more profound going on?

It turns to be no fluke. If the entries in the A matrix are real, and the initial condition \vec{x}_0 is real, then the solution to the differential equation $\frac{d}{dt}\vec{x} = A\vec{x}$ with $\vec{x}(0) = \vec{x}_0$ will also be real, *regardless of whether the eigenvalues of A are real, imaginary, or complex*. If a matrix $A \in \mathbf{R}^{n \times n}$ has some complex eigenvalues, then those eigenvalues will always arise in complex conjugate pairs. Furthermore, the eigenvectors associated to those eigenvalues arise on complex conjugate pairs. This will lead to the kind of cancellation that you saw in here, every single time.

After all, the quantities that we observe in the world are always purely real, so we would expect that the solutions to our models would also be real-valued.

6 RLC Responses: Critically Damped Case

Building on the previous problem, consider the following circuit with specified component values: (Notice R is not specified yet. You'll have to figure out what that is.)



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 1.

- a) For what value of R is there going to be a single eigenvalue of A ?

Solution

If the terms under the square root, i.e., the discriminant of the quadratic formula, is 0, then we have a single value. More concretely,

$$\frac{R^2}{L^2} - \frac{4}{LC} = 0 \tag{25}$$

$$\Rightarrow \frac{R^2}{L^2} = \frac{4}{LC} \tag{26}$$

$$\therefore R = 2\sqrt{\frac{L}{C}} \tag{27}$$

- b) Find the eigenvalues and eigenspaces of A . What are the dimensions of the corresponding eigenspaces? (i.e. how many linearly independent eigenvectors can you find associated with this eigenvalue?)

For this part, assume the given values for the capacitor and the inductor, as well as the critical value for the resistance R that you found in the previous part. It is easier to do the algebra with a non-symbolic matrix to work with.

Solution

Our system's matrix becomes,

$$A = \begin{bmatrix} -4 \times 10^6 & -4 \times 10^4 \\ 10^8 & 0 \end{bmatrix} \tag{28}$$

Our single eigenvector is,

$$\lambda = -\frac{R}{2L} = -2 \times 10^6 \quad (29)$$

Hence, the eigenvector is a basis of the nullspace of $A - \lambda I$,

$$\begin{bmatrix} -2 \times 10^6 & -4 \times 10^4 \\ 10^8 & 2 \times 10^6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (30)$$

Hence, the eigenvector, $\vec{v} = \alpha \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -50 \end{bmatrix}$. We have only one eigenvector, since we have a single dimensional nullspace.

- c) In the provided Jupyter notebook, move the sliders to the resistance value you found in the first part and $C = 10nF$. Sketch $V_c(t)$ and comment on its appearance. Additionally, sketch the location of the eigenvalues on the complex plane. What happens to the voltage and eigenvalues as you slightly increase or decrease R ?

Solution

At the $R = 100$, $V_c(t)$ appears to decay exponentially. A slight increase in R causes the voltage to decay more slowly. A slight decrease in R causes a voltage undershoot and eventually oscillations. The eigenvalues have converged into the same point at $(-2 \times 10^6, 0)$. Increasing R makes them split into two points, and both points remain on the real axis. One point goes towards the origin, while the other goes towards negative infinity. Decreasing R splits the eigenvalues back into their complex conjugates.

7 (OPTIONAL) Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very effective way to really learn material. Having some practice at trying to create problems helps you study for exams much better than simply solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really consolidate your understanding of the course material.

8 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

- a) **What sources (if any) did you use as you worked through the homework?**
- b) **If you worked with someone on this homework, who did you work with?** List names and student ID's. (In case of homework party, you can also just describe the group.)
- c) **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)
- d) **Roughly how many total hours did you work on this homework?**