This homework is due on Thursday, November 5, 2020, at 10:59PM.

Self-grades are due on Thursday, November 12, 2020, at 10:59PM.

1 SVD I

Find the singular value decomposition of the following matrix (leave all work in exact form, not decimal):

$$A = \begin{bmatrix} 2 & 2 \\ 3 & -3 \end{bmatrix}$$

a) Find the eigenvalues of $AA^\top$ and order them from largest to smallest, $\lambda_1 > \lambda_2$.

b) Find orthonormal eigenvectors $\vec{u}_i$ of $AA^\top$ (all eigenvectors are mutually orthogonal and unit length).

c) Find the singular values $\sigma_i = \sqrt{\lambda_i}$. Find the $\vec{v}_i$ vectors from:

$$A^\top \vec{u}_i = \sigma_i \vec{v}_i$$

d) Write out A as a weighted sum of rank-1 matrices:

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^\top + \sigma_2 \vec{u}_2 \vec{v}_2^\top$$

2 Rank 1 Decomposition

In this problem, we will decompose a few images into linear combinations of rank 1 matrices. Remember that outer product of two vectors $\vec{s}\vec{g}^T$ gives a rank 1 matrix. It has rank 1 because clearly, the column span is one-dimensional — multiples of $\vec{s}$ only — and the row span is also one dimensional — multiples of $\vec{g}^T$ only.

For example, if $\vec{s}$ and $\vec{g}$ are two vectors of dimension 5, then $\vec{s}\vec{g}^T$ is given as follows.

$$\vec{s} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{bmatrix} \qquad \vec{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \end{bmatrix}$$

$$\vec{s}\vec{g}^T = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{bmatrix} \begin{bmatrix} g_1 & g_2 & g_3 & g_4 & g_5 \end{bmatrix} = \begin{bmatrix} s_1g_1 & s_1g_2 & s_1g_3 & s_1g_4 & s_1g_5 \\ s_2g_1 & s_2g_2 & s_2g_3 & s_2g_4 & s_2g_5 \\ s_3g_1 & s_3g_2 & s_3g_3 & s_3g_4 & s_3g_5 \\ s_4g_1 & s_4g_2 & s_4g_3 & s_4g_4 & s_4g_5 \\ s_5g_1 & s_5g_2 & s_5g_3 & s_5g_4 & s_5g_5 \end{bmatrix}$$

a) Consider a standard 8×8 chessboard shown in Figure 1. Assume that black colors represent -1 and that white colors represent 1.

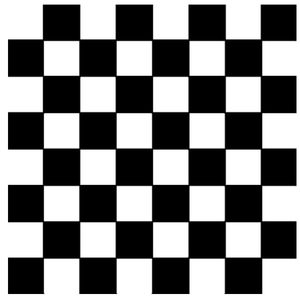

Figure 1: 8×8 chessboard.

Hence, that the chessboard is given by the following 8×8 matrix C_1:

$$C_1 = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

Express C_1 as a linear combination of outer products. *Hint: In order to determine how many rank 1 matrices you need to combine to represent the matrix, find the rank of the matrix you are trying to represent.*

b) For the same chessboard shown in Figure 1, now assume that black colors represent 0 and that white colors represent 1.

Hence, the chessboard is given by the following 8×8 matrix C_2:

$$C_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Express C_2 as a linear combination of outer products.

c) Now consider the Swiss flag shown in Figure 2. Assume that red colors represent 0 and that white colors represent 1.

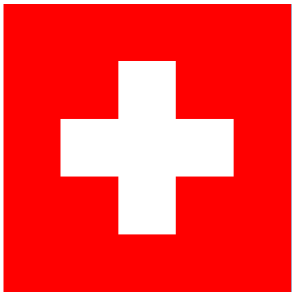

Figure 2: Swiss flag.

Assume that the Swiss flag is given by the following 5×5 matrix S:

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Furthermore, we know that the Swiss flag can be viewed as a superposition of the following pairs of images:

Figure 3: Pairs of images - Option 1

Figure 4: Pairs of images - Option 2

Express the S in two different ways: i) as a linear combination of the outer products inspired by the Option 1 images and ii) as a linear combination of outer products inspired by the Option 2 images.

3 SVD properties

In this question, we look at some properties of SVD. Below we consider a m by n matrix whose SVD writes $A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$ where r is the rank of the matrix.

a) We know that A can also be represented in matrix form as $A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T = U_1 S {V_1}^T$ where $U_1 = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r]$, $S = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$, and $V_1 = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r]$. Show that $\sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T = U_1 S {V_1}^T$. Note that the math does not assume any further property for U_1 and V_1 than that they are of compatible shape as long as S is diagonal.

   The full SVD of A writes $A = U\Sigma V^T$ where $U = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r, \vec{u}_{r+1}, \ldots, \vec{u}_m]$ and $V = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r, \vec{v}_{r+1}, \ldots, \vec{v}_n]$ are orthornormal matrices with the first r columns being the same as those of U' and V'.
$$\Sigma = \begin{bmatrix} S_{r\times r} & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}.$$

b) Suppose $r < \min(m, n)$, what is a basis of the null space of A? Prove your answer.

c) What is the basis of the range of A? Prove your answer.

d) Suppose $n = m = r$, that is, A is square and full rank. Find the inverse of A in terms of U, Σ, and V.

4 Induced Matrix Norms

Often, the general effect of matrices on their inputs is really hard to predict. To overcome this, we usually try to "bound" the effect a matrix has on input vectors. We will work through a simple case of bounding the output of an $n \times n$ matrix T given that T is diagonalizable and symmetric. We will consider the system
$$\vec{y} = T\vec{x},$$
where $\vec{x}$ is the input vector and $\vec{y}$ is the output vector.

a) Let $U = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n]$ be a set of orthonormal eigenvectors of the matrix T. **Decompose the generic input vector $\vec{x}$ into a linear combination of these eigenvectors.**

b) Let the eigenvalues of T be $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with $|\lambda_1| \geq |\lambda_2| \geq \ldots |\lambda_n|$. If we represent $\vec{x}$ as a linear combination of the eigenvectors of T, **what is the Euclidean norm of the output vector $\vec{y}$, $\|\vec{y}\|$?**
   Hint: You can use the fact that Euclidean norms are preserved with orthonormal transforms.

c) Say you do not know all the eigenvalues of T, but you know the largest eigenvalue λ_1. If the norm $||\vec{x}|| = \alpha$, **how big could $\|\vec{y}\|$ be**?

   The maximum factor by which a square matrix can grow the norm of a vector is called the induced norm for that matrix. Although we had you do the derivation above for a symmetric matrix T, the fact that $\|A\vec{x}\| = \sqrt{\vec{x}^T A^T A \vec{x}}$ can be used to show how to generalize this concept of induced norm for general matrices.

5 Closed-Loop Control of SIXT33N

To make our control more robust, we introduce feedback, turning our open-loop controller into a closed-loop controller. In this problem, we derive the closed-loop control scheme you will use to make SIXT33N reliably drive straight.

We introduce $\delta(k) = d_L(k) - d_R(k)$ as the difference in positions between the two wheels. We will consider a proportional control scheme, which introduces a feedback term into our input equation in which we apply gains k_L and k_R to $\delta(k)$ to modify our input at each timestep in an effort to prevent $|\delta(k)|$ from growing without bound. To do this, we will modify our inputs $u_L(k)$ and $u_R(k)$ to be:

$$\begin{aligned} u_L(k) &= \frac{v^* + \beta_L}{\theta_L} - k_L \frac{\delta(k)}{\theta_L} \\ u_R(k) &= \frac{v^* + \beta_R}{\theta_R} + k_R \frac{\delta(k)}{\theta_R} \end{aligned}$$

Substituting into the open-loop equations

$$\begin{aligned} d_L(k+1) - d_L(k) &= \theta_L u_L(k) - \beta_L \\ d_R(k+1) - d_R(k) &= \theta_R u_R(k) - \beta_R \end{aligned} \tag{1}$$

we obtain:

$$\begin{aligned} d_L(k+1) - d_L(k) &= v^* - k_L \delta(k) \\ d_R(k+1) - d_R(k) &= v^* + k_R \delta(k) \end{aligned} \tag{2}$$

a) Let's look a bit more closely at picking k_L and k_R. First, we need to figure out what happens to $\delta(k)$ over time. Find $\delta(k+1)$ in terms of $\delta(k)$.

b) Given your work above, what is the eigenvalue of the system defined by $\delta(k)$? For discrete-time systems like our system, $\lambda \in (-1, 1)$ is considered stable. Are $\lambda \in [0, 1)$ and $\lambda \in (-1, 0]$ identical in function for our system? Which one is "better"? (*Hint:* Preventing oscillation is a desired benefit.)

   Based on your choice for the range of λ above, how should we set k_L and k_R in the end?

c) Let's re-introduce the model mismatch in order to model environmental discrepancies, disturbances, etc. How does closed-loop control fare under model mismatch? Find $\delta_{ss} = \delta[k \to \infty]$, assuming that $\delta[0] = \delta_0$. What is δ_{ss}? (To make this easier, you may leave your answer in terms of appropriately defined c and λ obtained from an equation in the form of $\delta(k+1) = \delta(k)\lambda + c$.)

   Check your work by verifying that you reproduce the equation in part (c) if all model mismatch terms are zero. Is it better than the open-loop model mismatch?

$$\begin{aligned} d_L(k+1) - d_L(k) &= (\theta_L + \Delta\theta_L) u_L(k) - (\beta_L + \Delta\beta_L) \\ d_R(k+1) - d_R(k) &= (\theta_R + \Delta\theta_R) u_R(k) - (\beta_R + \Delta\beta_R) \end{aligned}$$

$$\begin{aligned} u_L(k) &= \frac{v^* + \beta_L}{\theta_L} - k_L \frac{\delta(k)}{\theta_L} \\ u_R(k) &= \frac{v^* + \beta_R}{\theta_R} + k_R \frac{\delta(k)}{\theta_R} \end{aligned}$$

6 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student!
We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?**
List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **Roughly how many total hours did you work on this homework?**

d) **Do you have any feedback on this homework assignment?**