This homework is due on Wednesday, March 11, 2020, at 11:59PM.

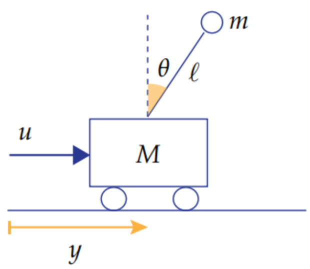
Self-grades are due on Monday, March 16, 2020, at 11:59PM.

1 Inverted Pendulum on a Rolling Cart (Mechanical)

Consider the inverted pendulum depicted below, which is placed on a rolling cart and whose equations of motion are given by:

$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2\ell\sin\theta - g\sin\theta\cos\theta\right)$$
$$\ddot{\theta} = \frac{1}{\ell(\frac{M}{m} + \sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2\ell\cos\theta\sin\theta + \frac{M+m}{m}g\sin\theta\right).$$

where we use $\dot{x}$ to denote the time derivative of x; that is, $\dot{y} = \frac{dy}{dt}$, $\dot{\theta} = \frac{d\theta}{dt}$, $\ddot{y} = \frac{d^2y}{dt^2}$ and $\ddot{\theta} = \frac{d^2\theta}{dt^2}$.



The problems below will prepare us for a future homework problem where we will design a control algorithm to stabilize the upright position.

a) Write the state model using the variables $x_1(t) = \theta(t)$, $x_2(t) = \dot{\theta}(t)$, and $x_3(t) = \dot{y}(t)$. We do not include $y(t)$ as a state variable because we are interested in stabilizing at the point $\theta = 0$, $\dot{\theta} = 0$, $\dot{y} = 0$, and we are not concerned about the final value of the position $y(t)$.

b) Show that $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ is an equilibrium point with $u = 0$.

c) Linearize this model at the equilibrium $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, and $u = 0$, and indicate the resulting A and B matrices.

2 Single-dimensional linearization

This is an exercise in linearizing a scalar system. The scalar nonlinear differential equation we have is

$$\frac{d}{dt}x(t) = \sin\big(x(t)\big) + u(t). \tag{1}$$

a) Find the equilibrium points for $u^* = 0$. You can do this by sketching $\sin(x)$ for $-4\pi \leq x \leq 4\pi$ and intersecting it with the horizontal line at 0. This will give you the equilibrium points x^* where $sin(x^*) + u^* = 0$.

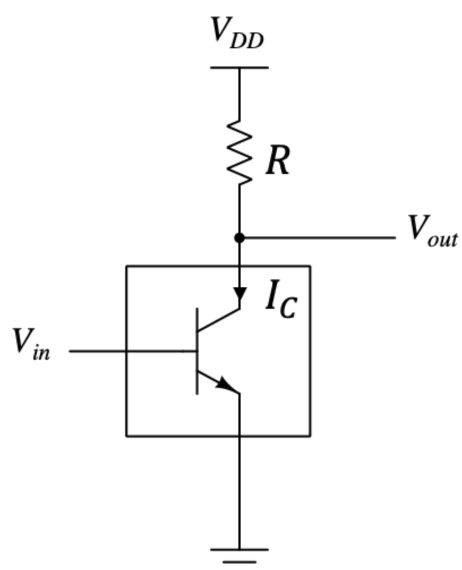
b) Linearize the system (1) around the equilibrium $(x_0^*, u^*) = (0, 0)$. **What is the resulting linearized scalar differential equation for $\tilde{x}(t) = x(t) - x_0^* = x(t) - 0$, involving $\tilde{u}(t) = u(t) - u^* = u(t) - 0$?**

c) For the linearized approximate system model that you found in the previous part, what happens if we try to discretize time to intervals of duration T? Assume now we use a piecewise constant control input over duration T, that T is small relative to the ranges of controls applied, and that we sample the state x every T (that is, at every $t = nT$, where n is an integer) as well. **Write out the resulting scalar discrete-time control system model.** This model is an approximation of what will happen if we actually applied a piecewise constant control input to the original nonlinear differential equation.

3 Linearizing for understanding amplification

Linearization isn't just something that is important for control, robotics, machine learning, and optimization — it is one of the standard tools for circuits

The circuit below is a voltage amplifier, where the element inside the box is a bipolar junction transistor (BJT).



The bipolar transistor in the circuit can be modeled quite accurately as a nonlinear, voltage-controlled current source, where the collector current I_C is given by

$$I_C(V_{in}) = I_S e^{\frac{V_{in}}{V_{TH}}} \tag{2}$$

where V_{TH} is the thermal voltage. We can assume $V_{TH} = 26$ mV at temperatures of 300K (close to room temperature). I_S is a constant whose exact value we are not giving you because we want you to find ways of eliminating it in favor of other quantities whenever possible.

With this amplifier, small variations in the input voltage V_{in} can turn into large variations in the output voltage V_{out} under the right conditions. We're going to investigate this amplification using linearization.

Let's consider the 2N3904 transistor, where the above expression for $I_C(V_{in})$ holds as long as $0.2\text{V} < V_{out} < 40\text{V}$, and $0.1\text{mA} < I_C < 10\text{mA}$.

(Note that the 2N3904 is a cheap transistor that people often use in personal projects. You can get them for 3 cents each if you buy in bulk.)

a) **Write a symbolic expression for V_{out} as a function of I_C.**

b) Now let's linearize I_C in the neighborhood of an input voltage V_{in}^* and a specific I_C^*. Assume that you have a found a particular pair of input voltage V_{in}^* and current I_C^* that satisfy the current equation (2).

We can look at nearby input voltages and see how much the current changes. We can write the linearized expression for the collector current around this point as:

$$I_C(V_{in}) = I_C(V_{in}^*) + \delta I_C \approx I_C^* + m(V_{in} - V_{in}^*) = I_C^* + m\ \delta V_{in} \tag{3}$$

where $\delta V_{in} = V_{in} - V_{in}^*$ is the change in input voltage and $\delta I_C = I_C - I_C^*$ is the change in collector current.

What is m here as a function of I_C^* and V_{TH}?

(If you take EE105, you will learn that this m is called the transconductance, which is usually written g_m, and is the single most important parameter in most analog circuit designs.)

(HINT: First just find m by taking the appropriate derivative and using the chain rule as needed. Then leverage the special properties of the exponential function to express it in terms of the desired quantities.)

c) We now have a linear relationship between small changes in current and voltage, $\delta I_C = m\ \delta V_{in}$ around a known solution (I_C^*, V_{in}^*). This is called a "bias point" in circuits terminology.

Going back to your equation from part (a), plug in your linearized equation for I_C. Define the appropriate V_{out}^* so that it makes sense to view $V_{out} = V_{out}^* + \delta V_{out}$ when we have $V_{in} = V_{in}^* + \delta V_{in}$, and **find the approximate linear relationship between δV_{out} and δV_{in}.**

The ratio $\frac{\delta V_{out}}{\delta V_{in}}$ is called the small-signal voltage gain of this amplifier around this bias point.

d) Assuming that $V_{DD} = 10V$, $R = 1k\Omega$, and $I_C^* = 1mA$ when $V_{in}^* = 0.65V$, **what is the small-signal voltage gain $\frac{\delta V_{out}}{\delta V_{in}}$, between the input and the output around this bias point?** (one or two digits of precision is plenty)

e) If $I_C^* = 9mA$ when $V_{in}^* = 0.7V$, **what is the small-signal voltage gain around this bias point?** (one or two digits is plenty)

4 (OPTIONAL) Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very effective way to really learn material. Having some practice at trying to create problems helps you study for exams much better than simply solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really consolidate your understanding of the course material.

5 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?** List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)

d) **Do you have any feedback on this homework assignment?**

e) **Roughly how many total hours did you work on this homework?**