This homework is due on Wednesday, February 19, 2020, at 11:59PM.

Self-grades are due on Monday, February 24, 2019, at 11:59PM.

1 Complex Numbers

A common way to visualize complex numbers is to use the complex plane. Recall that a complex number z is often represented in Cartesian form.

$$z = x + jy \text{ with } \mathrm{Re}\{z\} = x \text{ and } \mathrm{Im}\{z\} = y$$

See Figure 1 for a visualization of z in the complex plane.

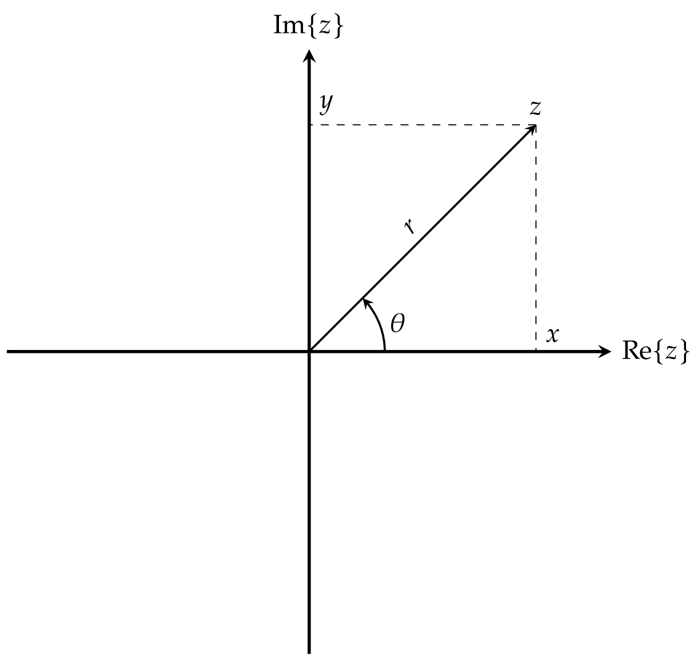


Figure 1: Complex Plane

In this question, we will derive the polar form of a complex number and use this form to make some interesting conclusions.

a) **Calculate the length of z in terms of x and y as shown in Figure 1.** This is the magnitude of a complex number and is denoted by $|z|$ or r.

   (Hint: Use the Pythagorean theorem.)

b) **Represent x, the real part of z, and y, the imaginary part of z, in terms of r and θ.**

c) **Substitute for x and y in z.** Use Euler's identity[1] $e^{j\theta} = \cos\theta + j\sin\theta$ to **conclude that,**

$$z = re^{j\theta}.$$

d) In the complex plane, **sketch the set of all the complex numbers such that $|z| = 1$. What are the z values where the sketched figure intersects the real axis and the imaginary axis?**

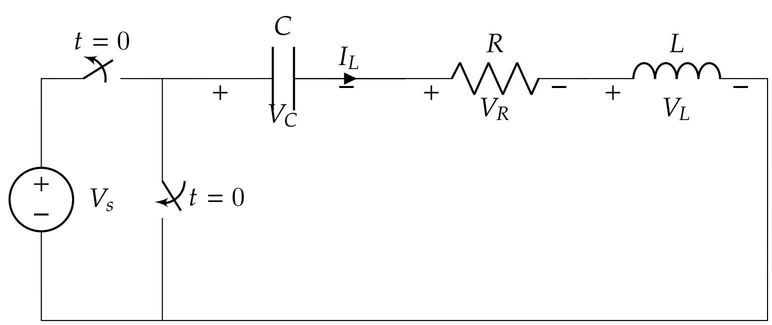
e) If $z = re^{j\theta}$, **prove that $\bar{z} = re^{-j\theta}$**. Recall that the complex conjugate of a complex number $z = x + jy$ is $\bar{z} = x - jy$.

f) **Show (by direct calculation) that,**

$$r^2 = z\bar{z}.$$

2 RLC Responses: Initial Part

Consider the following circuit like you saw in lecture:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

In this problem, the current through the inductor and the voltage across the capacitor are the natural physical state variables since these are what correlate to how energy is actually stored in the system. (A magnetic field through the inductor and an electric field within the capacitor.)

a) **Write the system of differential equations in terms of state variables $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ that describes this circuit for $t \geq 0$. Leave the system symbolic in terms of V_s, L, R, and C.**

[1] also known as de Moivre's Theorem.

b) Write the system of equations in vector/matrix form with the vector state variable $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. This should be in the form $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ with a 2×2 matrix A.

c) Find the eigenvalues of the A matrix symbolically.
(Hint: the quadratic formula will be involved.)

d) Under what condition on the circuit parameters R, L, C are there going to be a pair of distinct purely real eigenvalues of A?

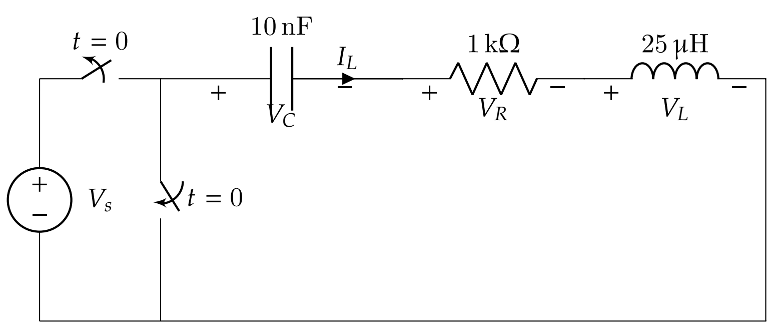
e) Under what condition on the circuit parameters R, L, C are there going to be a pair of purely imaginary eigenvalues of A?

f) Assuming that the circuit parameters are such that there are a pair of (potentially complex) eigenvalues λ_1, λ_2 so that $\lambda_1 \neq \lambda_2$, find eigenvectors $\vec{v}_{\lambda_1}, \vec{v}_{\lambda_2}$ corresponding to them.
(HINT: Rather than trying to find the relevant nullspaces, etc., try to find eigenvectors of the form $\begin{bmatrix} 1 \\ y \end{bmatrix}$ where we just want to find the missing entry y. Can you see from the structure of the A matrix why we might want to try that guess?)

g) Assuming circuit parameters such that the two eigenvalues of A are distinct, let $V = [\vec{v}_{\lambda_1}, \vec{v}_{\lambda_2}]$ be a specific eigenbasis. Consider a coordinate system for which we can write $\vec{x}(t) = V\vec{\widetilde{x}}(t)$. What is the $\widetilde{A}$ so that $\frac{d}{dt}\vec{\widetilde{x}}(t) = \widetilde{A}\vec{\widetilde{x}}(t)$? It is fine to have your answer expressed symbolically using λ_1, λ_2.

3 RLC Responses: Overdamped Case

Building on the previous problem, consider the following circuit with specified component values:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2.

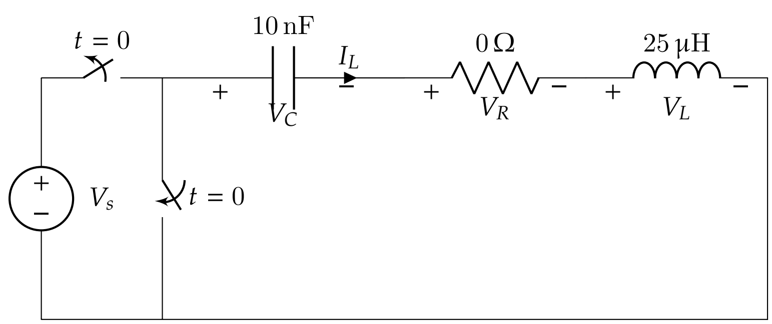
a) Suppose $R = 1\,\mathrm{k}\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1$ Volt. Find the initial conditions for $\vec{\tilde{x}}(0)$. Recall that $\vec{\tilde{x}}$ is in the changed "nice" eigenbasis coordinates from the first problem.

b) Continuing the previous part, find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$.

c) In the provided Jupyter notebook, move the sliders to approximately $R = 1k\Omega$ and $C = 10nF$. Sketch $V_c(t)$ and comment on its appearance. Additionally, sketch the location of the eigenvalues on the complex plane.

4 RLC Responses: Undamped Case

Building on the previous problem, consider the following circuit with specified component values:



Assume that the capacitor is charged to V_s and there is no current in the inductor for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2.

a) Suppose $R = 0\,\mathrm{k}\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1$ Volt. Find the initial conditions for $\vec{\tilde{x}}(0)$. Recall that $\vec{\tilde{x}}$ is in the changed "nice" eigenbasis coordinates from the first problem.

b) Continuing the previous part, find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$.

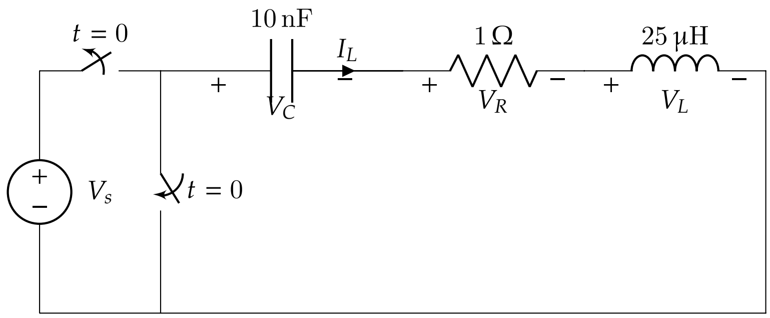
c) In the provided Jupyter notebook, move the sliders to approximately $R = 0\Omega$ and $C = 10nF$. Sketch $V_c(t)$ and comment on its appearance. Additionally, sketch the location of the eigenvalues on the complex plane.

Are the waveforms for $x_1(t)$ and $x_2(t)$ "transient" — do they die out with time?

Note: Because there is no resistance, this is called the "undamped" case.

5 RLC Responses: Underdamped Case

Building on the previous problem, consider the following circuit with specified component values:



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 2. You may round numbers to make the algebra more simple.

a) Now suppose that $R = 1\,\Omega$ and the other component values are as specified in the circuit. Assume that $V_s = 1$ Volt. Find the initial conditions for $\vec{\tilde{x}}(0)$. Recall that $\vec{\tilde{x}}$ is in the changed "nice" eigenbasis coordinates from the first problem.

b) Continuing the previous part, find $x_1(t) = I_L(t)$ and $x_2(t) = V_C(t)$ for $t \geq 0$.

(HINT: Remember that $e^{a+jb} = e^a e^{jb}$.)

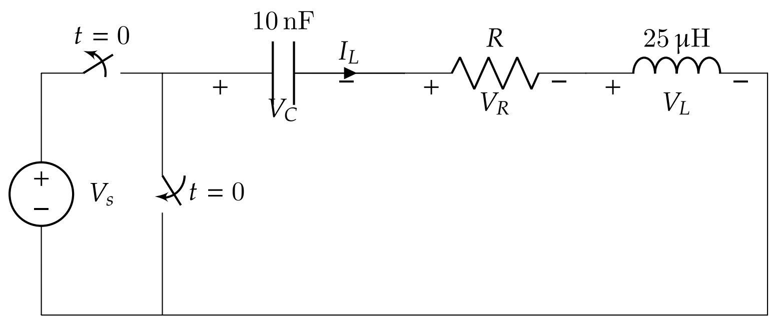
c) In the provided Jupyter notebook, move the sliders to approximately $R = 1\Omega$ and $C = 10nF$. Sketch $V_c(t)$ and comment on its appearance. Additionally, sketch the location of the eigenvalues on the complex plane. Are the waveforms for $x_1(t)$ and $x_2(t)$ "transient" — do they die out with time?

Note: Because the resistance is so small, this is called the "underdamped" case. It is good to reflect upon these waveforms to see why engineers consider such behavior to be reflective of systems that don't have enough damping.

d) Notice that you got answers in terms of complex exponentials. Why did the final voltage and current waveforms end up being purely real?

6 RLC Responses: Critically Damped Case

Building on the previous problem, consider the following circuit with specified component values: (Notice R is not specified yet. You'll have to figure out what that is.)



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 1.

a) For what value of R is there going to be a single eigenvalue of A?

b) Find the eigenvalues and eigenspaces of A. What are the dimensions of the corresponding eigenspaces? (i.e. how many linearly independent eigenvectors can you find associated with this eigenvalue?)

   For this part, assume the given values for the capacitor and the inductor, as well as the critical value for the resistance R that you found in the previous part. It is easier to do the algebra with a non-symbolic matrix to work with.

c) In the provided Jupyter notebook, move the sliders to the resistance value you found in the first part and $C = 10nF$. Sketch $V_c(t)$ and comment on its appearance. Additionally, sketch the location of the eigenvalues on the complex plane. What happens to the voltage and eigenvalues as you slightly increase or decrease R?

7 (OPTIONAL) Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very effective way to really learn material. Having some practice at trying to create problems helps you study for exams much better than simply solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really consolidate your understanding of the course material.

8 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?** List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)

d) **Roughly how many total hours did you work on this homework?**