1 Existence and uniqueness of solutions to differential equations

Let's show that if any function x satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \alpha x(t) \tag{1}$$

as well as

$$x(0) = x_0, \tag{2}$$

then it is unique: if y is *any function* that meets these two criteria then $x = y$.

In order to do this, we will first verify that a solution exists. Then we will compare it to a hypothetical alternative solution—and our goal will to be establish that these two solutions are equal.

a) **Verify that** $x_d(t) = x_0 e^{\alpha t}$ **satisfies** (1) **and** (2)**.** (For this proof, x_d will be the "reference solution" against which alternates will be compared.)

b) To show that this solution is in fact unique, we need to consider a hypothetical $y(t)$ that also satisfies (1) and (2).

Our goal is to show that $y(t) = x(t)$ for all $t \geq 0$. (The domain $t \geq 0$ is where we have defined the conditions (1) and (2). Outside of that domain, we don't have any constraints.)

How can we show that two things are equal? In the past, you have probably shown that two quantities or functions are equal by starting with one of them, and then manipulating the expression for it using valid substitutions and simplifications until you get the expression for the other one. However, here, we don't have an expression for $y(t)$ so that style of approach won't work.

In such cases, we basically have a couple of basic ways of showing that two things are the same.

- Take the difference of them, and somehow argue that it is 0.
- Take the ratio of them, and somehow argue that it is 1.

We will follow the ratio approach in this problem. First assume that $x_0 \neq 0$. In this case, we are free to define $z(t) = \frac{y(t)}{x_d(t)}$ since we are dividing by something other than zero.

What is $z(0)$**?**

c) **Take the derivative** $\frac{d}{dt}z(t)$ **and simplify using** (1) **and what you know about the derivative of** $x_d(t)$**.**

(HINT: The quotient rule for differentiation might be helpful since a ratio is involved.)

You should see that this derivative is always 0 and hence $z(t)$ does not change. **What does that imply for** y **and** x_d **?**

d) At this point, we have shown uniqueness in most cases. Just one special case is left: $x_0 = 0$. The ratio technique omitted this case, because as $x_d(t) = 0$, x_d cannot be the denominator of a fraction.

To complete our proof we must to show that if $x_0 = 0$, then $y(t) = 0$ for all t, and we will do so by assuming that $y(t)$ is not identically 0 for $t > 0$—that is, at some $t_0 > 0$ $y(t_0) = k \neq 0$.

From (2), we know that $y(0) = 0$. In this part, we will try to work backwards in time from the point $t = t_0$ to $t = 0$ and conclude that y violates (2).

Apply the change of variables $t = t_0 - \tau$ to (1) to get a new differential equation for $\widetilde{x}(\tau) = x(t_0 - \tau)$ that specifies how $\frac{d}{d\tau}\widetilde{x}(\tau)$ must relate to $\widetilde{x}(\tau)$. This should hold for $-\infty < \tau \leq t_0$.

e) Because the previous part resulted in a differential equation of a form for which we have already proved uniqueness for the case of nonzero initial condition, and since $\widetilde{y}(0) = y(t_0) = k \neq 0$, we know what $\widetilde{y}(\tau)$ must be. **Write the expressions for $\widetilde{y}(\tau)$ for $\tau \in [0, t_0]$ and what that implies for $y(t)$ for $t \in [0, t_0]$.**

f) **Evaluate $y(0)$ and argue that this is a contradiction for the specified initial condition** (2).

Consequently, such a $y(t)$ cannot exist and only the all zero solution is permitted — establishing uniqueness in this case of $x_0 = 0$ as well.

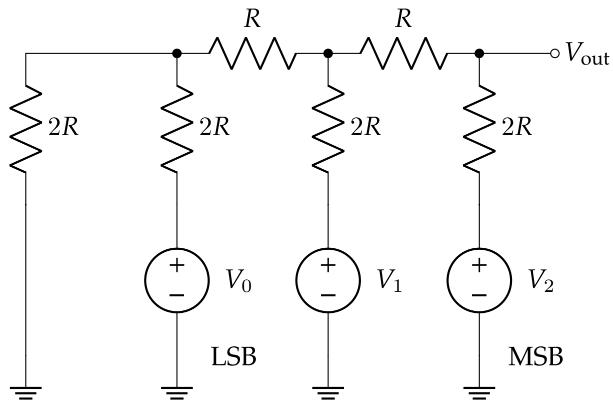
g) Explain in your own words why it matters that solutions to these differential equations are unique.

Although we gave you lots of guidance in this problem, we hope that you can internalize this way of thinking.

This elementary approach to proving the uniqueness of solutions to differential equations works for the kinds of linear differential equations that we will tend to encounter in EE16B. For more complicated nonlinear differential equations, further conditions are required for uniqueness (appropriate continuity and differentiability) and proofs can be found in upper-division mathematics courses on differential equations when you study the Picard-Lindelöf theorem. (It involves looking at the magnitude of the difference of the two hypothetical solutions and showing this has to be arbitrarily small and hence zero. However, the basic elementary case we have established here can be viewed as a building block — the quotient rule gets invoked in the appropriate place, etc. The additional ingredients that are out-of-scope for lower-division courses are fixed-point theorems — which you can think of as more general siblings of the intermediate-value theorem you saw in basic calculus.)

2 Digital-Analog Converter

A digital-analog converter (DAC) is a circuit for converting a digital representation of a number (binary) into a corresponding analog voltage. In this problem, we will consider a DAC made out of resistors only (resistive DAC) called the R-$2R$ ladder. Here is the circuit for a 3-bit resistive DAC.



Let $b_0, b_1, b_2 = \{0, 1\}$ (that is, either 1 or 0), and let the voltage sources $V_0 = b_0 V_{\text{DD}}$, $V_1 = b_1 V_{\text{DD}}$, $V_2 = b_2 V_{\text{DD}}$, where V_{DD} is the supply voltage.

As you may have noticed, (b_2, b_1, b_0) represents a 3-bit binary (unsigned) number where each of b_i is a binary bit. We will now analyze how this converter functions.

a) If $b_2, b_1, b_0 = 1, 0, 0$, what is V_{out}? Express your answer in terms of V_{DD}.

b) If $b_2, b_1, b_0 = 0, 1, 0$, what is V_{out}? Express your answer in terms of V_{DD}.

c) If $b_2, b_1, b_0 = 0, 0, 1$, what is V_{out}? Express your answer in terms of V_{DD}.

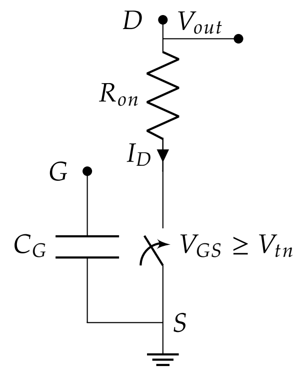
d) If $b_2, b_1, b_0 = 1, 1, 1$, what is V_{out}? Express your answer in terms of V_{DD}.

e) Finally, solve for V_{out} in terms of V_{DD} and the binary bits b_2, b_1, b_0.

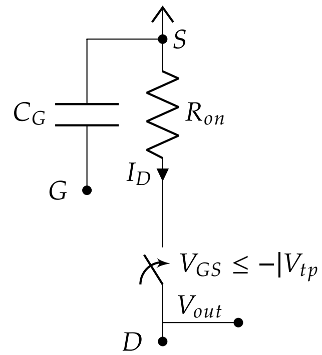
f) Explain how your results above show that the resistive DAC converts the 3-bit binary number (b_2, b_1, b_0) to the output analog voltage V_{out}.

3 Transistor Switch Model

We can improve our resistor-switch model of the transistor by adding in a gate capacitance. In this model, the gate capacitance C_G represents the lumped physical capacitance present on the gate node of all transistor devices. This capacitance is important as it determines the delay of a transistor logic chain.



(a) NMOS Transistor Resistor-switch-capacitor model



(b) PMOS Transistor Resistor-switch-capacitor model. Note we have drawn this so that it aligns with the inverter.

You have two CMOS inverters made from NMOS and PMOS devices. Both NMOS and PMOS devices have an "on resistance" of $R_{on} = 1$ kΩ, and each has a gate capacitance (input capacitance) of $C_G = 1$fF (femto-Farads = 10^{-15}). We assume the "off resistance" (the resistance when the transistor is off) is infinite (i.e. , the transistor acts as an open circuit when off). The supply voltage V_{DD} is 1V. The two inverters are connected in series, with the output of the first inverter driving the input of the second inverter (fig. 4).

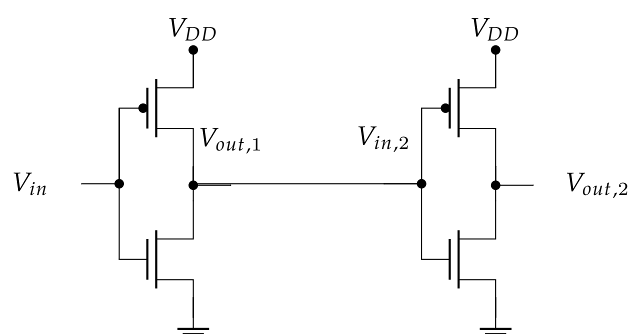


Figure 2: CMOS Inverter chain

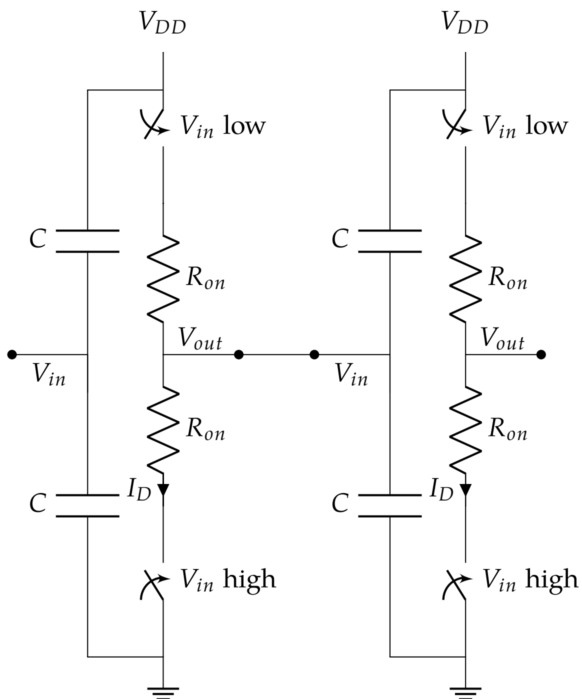


Figure 3: Inverter Transistor Resistor-switch model

a) Assume the input to the first inverter has been low ($V_{in} = 0$ V) for a long time, and then switches at time $t = 0$ to high ($V_{in} = V_{DD}$). **Draw a simple RC circuit and write a differential equation describing the output voltage of the first inverter ($V_{out,1}$) for time $t \geq 0$.** Don't forget that the second inverter is "loading" the output of the first inverter — you need to think about both of them.

b) Given the initial conditions in part (a), **solve for $V_{out,1}(t)$.**

c) **Sketch the output voltage of the first inverter, showing clearly (1) the initial value, (2) the initial slope, (3) the asymptotic value, and (4) the time that it takes for the voltage to decay to roughly 1/3 of its initial value.**

d) A long time later, the input to the first inverter switches low again.
Solve for $V_{out,1}(t)$.
Sketch the output voltage of the first inverter ($V_{out,1}$), showing clearly (1) the initial value, (2) the initial slope, and (3) the asymptotic value.

e) For each complete input cycle described above ($V_{in} = 0\text{V} \rightarrow 1\text{V} \rightarrow 0\text{V}$), **how much charge is pulled out of the power supply?** Give both a symbolic and numerical answer. **Consider only the charge needed to charge up the $V_{out,1}$ node.**

4 RC Circuit

Consider the circuit below, assume that when $t \leq 0$, the capacitor has no charge stored ($V_c(t = 0) = 0$). At $t = 0$, the switch closes. Assume that $V_s = 5\,\mathrm{V}$, $R = 100\,\Omega$, and $C = 10\,\mu\mathrm{F}$.

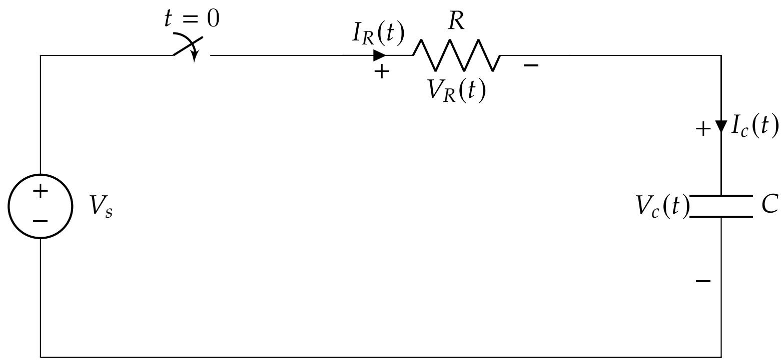


Figure 4: RC Circuit with Voltage Source

a) What are the boundary conditions for $I_c(t)$ (i.e. what are $I_c(t = 0)$ and $I_c(t \to \infty)$?)

b) Use KVL and the relationship between charge and current ($q = \int I dt$) to find the first order differential equation in terms of the current through the capacitor, I_c. Assume that $\frac{dV_s}{dt} = 0$. (*Hint:* You will need to take a derivative with respect to time to get the equation.)

c) What is the eigenvalue λ of this equation?

d) Using the eigenvalue and boundary conditions found in the previous parts, find an expression for $I_c(t)$ in terms of V_s, R, and C.

e) On what order of magnitude of time (nanoseconds, milliseconds, 10's of seconds, etc.) does this circuit settle (I_c is $< 5\%$ of its initial value)?

f) Give 2 ways to reduce the settling time of the circuit if we are allowed to change one component in the circuit.

g) Sketch the current vs. time plot of $I_c(t)$. Make sure to label $I_c(t)$ at $t = 0$, $t = \tau$, $t = 2\tau$, and $t = 3\tau$.

5 (OPTIONAL) Write Your Own Question And Provide a Thorough Solution.

Writing your own problems is a very effective way to really learn material. Having some practice at trying to create problems helps you study for exams much better than simply solving existing practice problems. This is because thinking about how to create an interesting problem forces you to really consolidate your understanding of the course material.

6 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?** List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)

d) **Roughly how many total hours did you work on this homework?**