Midterm Redo

The midterm redo is due on Monday, March 28, 2022, at 11:59PM.

The midterm redo is an optional assignment. However, if you want to qualify for the clobber policy, it is mandatory. As a reminder, the clobber policy grants the following:

```
midterm % after clobber = max(final % - 5%, original midterm %)
```

What does it mean to "complete" the midterm redo assignment? We'd like to see you make an honest attempt on every problem. You will most definitely qualify for the clobber then.

1. Complex Numbers

You are given the graph in Figure 1.

In this problem, you may use the $\text{atan2}(b, a)$ function to compute the angle (phase) for the complex number $a + jb$ as necessary.

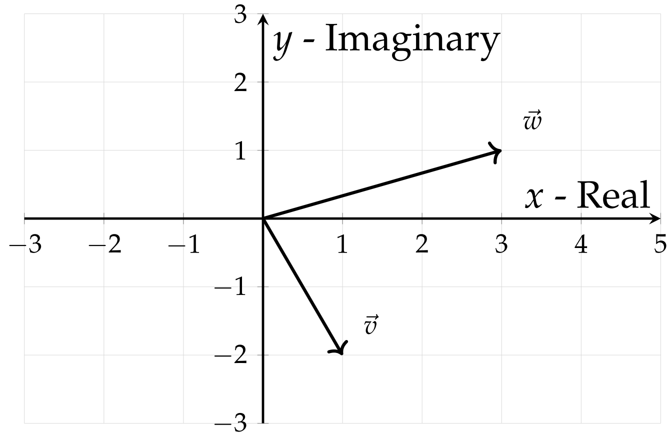


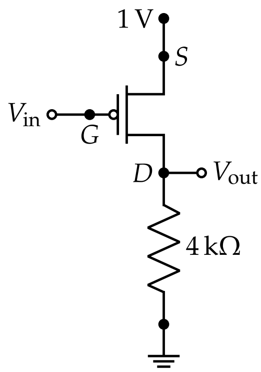
Figure 1: Vectors in the $\mathbf{x} - \mathbf{y}$ plane

(a) **What are the Cartesian ((x, y)) and Polar ($re^{j\theta}$) coordinates of $\vec{v}$?**

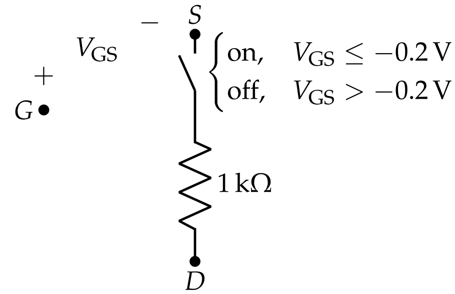
(b) **What are the Cartesian ((x, y)) and Polar ($re^{j\theta}$) coordinates of $\vec{w}$?**

2. PMOS Transistor Inverter

Consider the following schematic and PMOS model.



(a) A PMOS transistor circuit

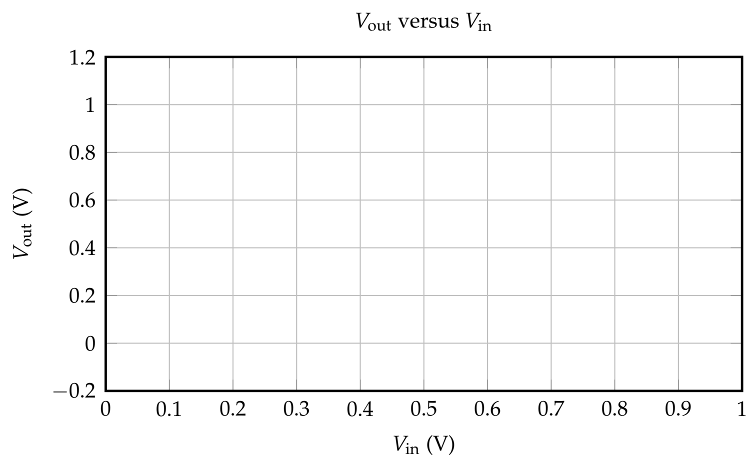


(b) Resistor and switch model for PMOS transistor.

Figure 2: PMOS figures.

Please plot the output V_{out} for the input V_{in} ranging from $0\,V$ **to** $1\,V$. *Justify your answer.*

NOTE: The y-axis ticks starts from $-0.2\,V$.

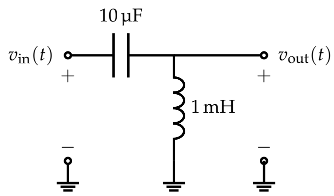


(I) V_{out} versus V_{in} Graph

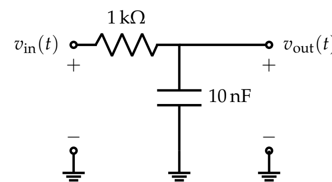
3. Filter Circuits

(a) Below, you have filter circuits A, B, C, D, each with specific component values. **Fill in the bubbles to match each filter to its corresponding magnitude transfer function plot out of choices I, II, III, IV.** Note that each plot may be assigned to filters **once, more than once, or not at all.** Each filter has **exactly one** corresponding plot.

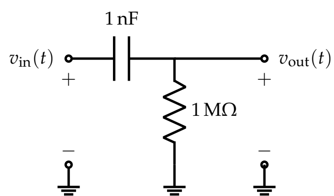
SI Prefixes and Exponent definitions: nano (n): 10^{-9}; micro (μ): 10^{-6}; milli (m): 10^{-3}; kilo (k): 10^{3}; mega (M): 10^{6}; giga (G): 10^{9}



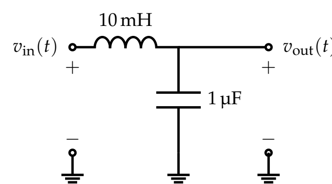
(A) Filter A.



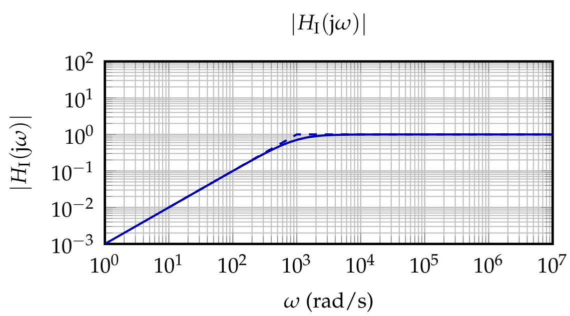
(B) Filter B.



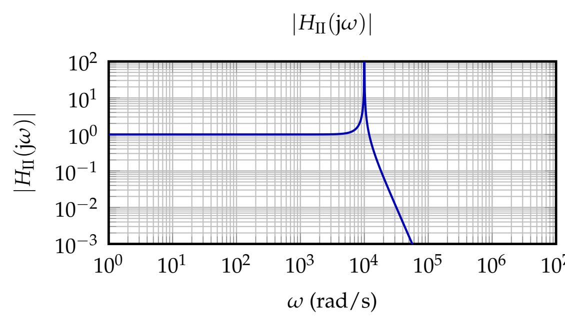
(C) Filter C.



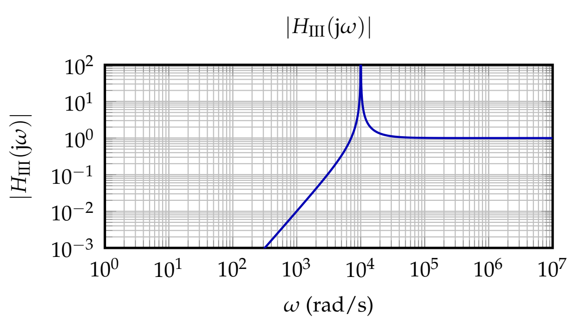
(D) Filter D.



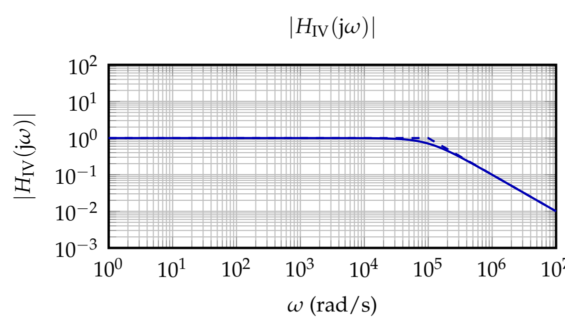
(I) Plot I.



(II) Plot II.



(III) Plot III.

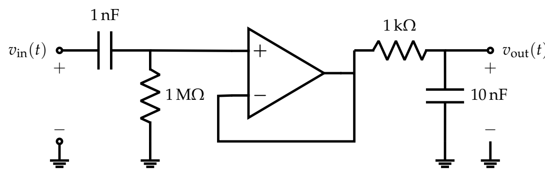


(IV) Plot IV.

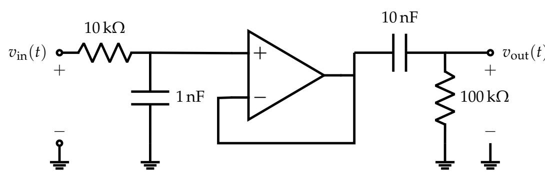
Filter Letter	Plot I	Plot II	Plot III	Plot IV
A	◯	◯	◯	◯
B	◯	◯	◯	◯
C	◯	◯	◯	◯
D	◯	◯	◯	◯

(b) Now, in order to design a band-pass filter, one possible way is to cascade two filters above. Below, you have filter circuits A, B, C, each with specific component values. **Fill in the bubbles to match each filter to its corresponding magnitude transfer function plot out of choices I, II, III.**

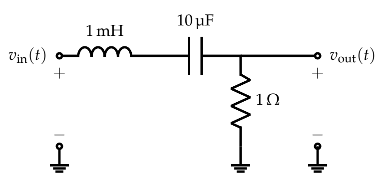
Note that each plot may be assigned to filters **once, more than once, or not at all.** Each filter has **exactly one** corresponding plot.



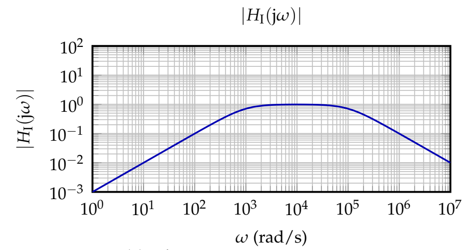
(A) Filter A.



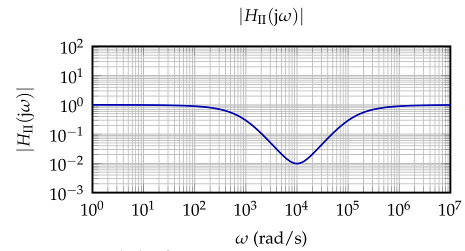
(B) Filter B.



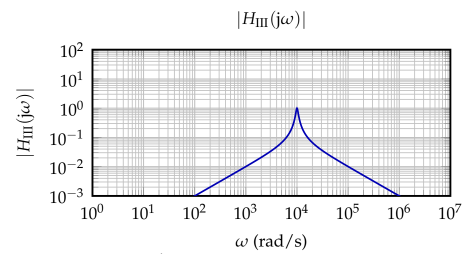
(C) Filter C.



(I) Plot I.



(II) Plot II.



III Plot III.

Filter Letter	Plot I	Plot II	Plot III
A	◯	◯	◯
B	◯	◯	◯
C	◯	◯	◯

4. Magnitude, Phase, and Cascades

Suppose you have the transfer function $H(\mathrm{j}\omega)$ for a system as given in eq. (1) below:

$$H(\mathrm{j}\omega) = \frac{20}{1 + \mathrm{j}\frac{\omega}{\omega_0}} \tag{1}$$

where $\omega_0 := 1 \times 10^7 \,\frac{\mathrm{rad}}{\mathrm{s}}$.

Answer the following questions.

(a) **What is the transfer function's magnitude $|H(\mathrm{j}\omega)|$ at $\omega = 0 \,\frac{\mathrm{rad}}{\mathrm{s}}$?**

(b) **What is the transfer function's magnitude $|H(\mathrm{j}\omega)|$ at $\omega = \infty \frac{\mathrm{rad}}{\mathrm{s}}$?**

(c) **What is the transfer function's phase $\measuredangle H(\mathrm{j}\omega)$ at $\omega = 1 \times 10^7 \,\frac{\mathrm{rad}}{\mathrm{s}}$?**

(d) You cascade the systems S_1 as defined by the transfer function:

$$H_1(\mathrm{j}\omega) = \frac{20}{1 + \mathrm{j}\frac{\omega}{\omega_1}} \tag{2}$$

with another system S_2 as defined by the transfer function:

$$H_2(\mathrm{j}\omega) = \frac{100}{1 + \mathrm{j}\frac{\omega}{\omega_2}}. \tag{3}$$

where $\omega_1 := 1 \times 10^7 \,\frac{\mathrm{rad}}{\mathrm{s}}$ and $\omega_2 := 1 \times 10^4 \,\frac{\mathrm{rad}}{\mathrm{s}}$.

You place S_2 after S_1, with unity-gain buffers in between. **Write the overall transfer function $H_{\mathrm{cascade}}(\mathrm{j}\omega)$ in terms of** $\mathrm{j}\omega$**.** *You do not need to simplify your answer for this subpart.*

5. Stability of Discrete-Time System

Suppose we are working with a linear model which has the form:

$$\underbrace{\begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix}}_{\vec{x}[i+1]} = \underbrace{\begin{bmatrix} 0 & 1 \\ k & 1 \end{bmatrix}}_{A_d} \underbrace{\begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix}}_{\vec{x}[i]} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w[i], \tag{4}$$

where $k \in \mathbb{R}$ is an unknown variable.

(a) **Give the range of k, such that the matrix A_d has only real eigenvalues**. *Justify your answer.*

(b) **Choose the possible k value(s) from the following options such that the above model is stable.**
Select all choices that apply.

i. $k = -\frac{3}{4}$.
ii. $k = 1$.
iii. $k = -\frac{1}{4}$.
iv. $k = \frac{1}{4}$.

6. Controllability and Eigenvalue Placement

Suppose we are working with a linear model with two-dimensional state $\vec{x}\colon \mathbb{N} \to \mathbb{R}^2$ but one-dimensional input $u\colon \mathbb{N} \to \mathbb{R}$:

$$\underbrace{\begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix}}_{=\vec{x}[i+1]} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{=A} \underbrace{\begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix}}_{=\vec{x}[i]} + \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{=\vec{b}} u[i] \tag{5}$$

where $b_1 \neq 0$ and $b_2 \neq 0$.

(a) **Show that the model in Equation (5) is not controllable.**

(b) Suppose we add feedback control of the form

$$u[i] := \underbrace{\begin{bmatrix} f_1 & f_2 \end{bmatrix}}_{=\vec{f}^\top} \underbrace{\begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix}}_{=\vec{x}[i]}. \tag{6}$$

Show that one of the eigenvalues of $A_{\mathrm{CL}} := A + \vec{b}\vec{f}^\top$ is 1, regardless of the values of f_1 and f_2.

7. Brain Stimulation

For his neuron-modeling project Krishna thought of consulting his close friend Radhika, who is a neuroscientist. According to Radhika's suggestions, Krishna came up with the following model of the cell-membrane of a neuron:

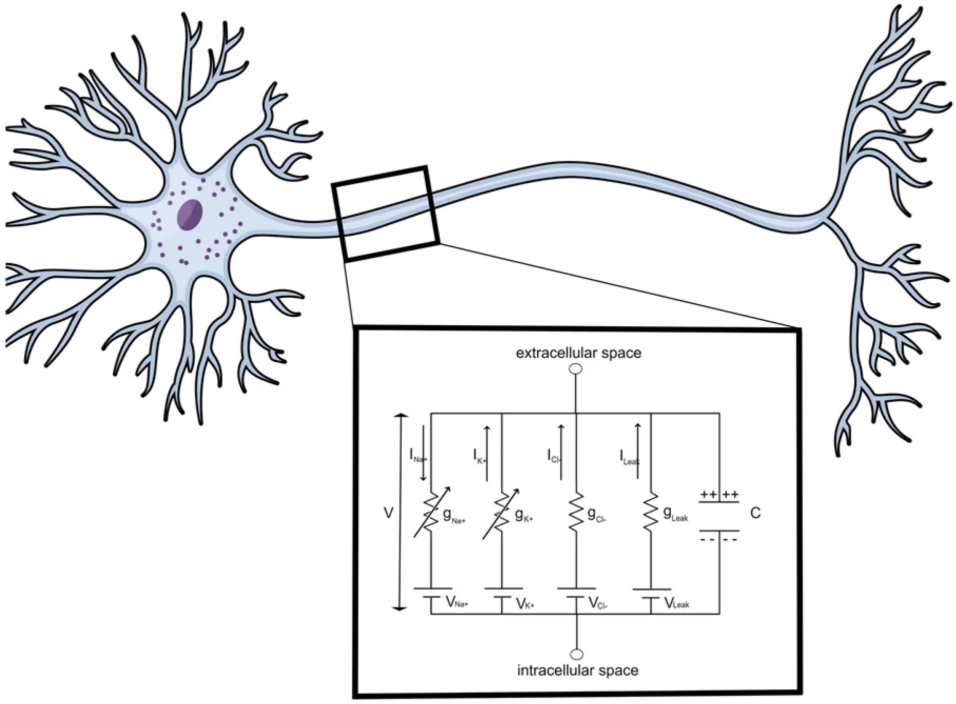


Figure 7: Electrical model of the neuron membrane

(a) Now Krishna wants to see how the neuron behaves to an external current stimulus. As he found the complete model very difficult to analyze, he starts his analysis with the following simple model:

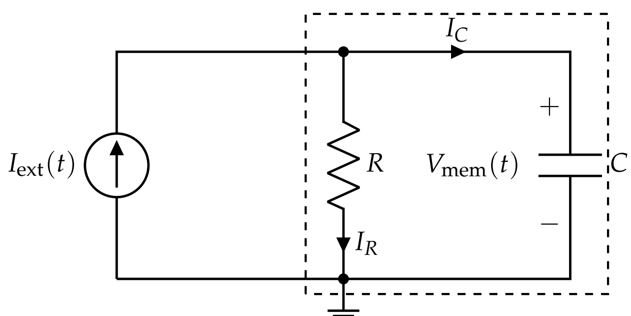
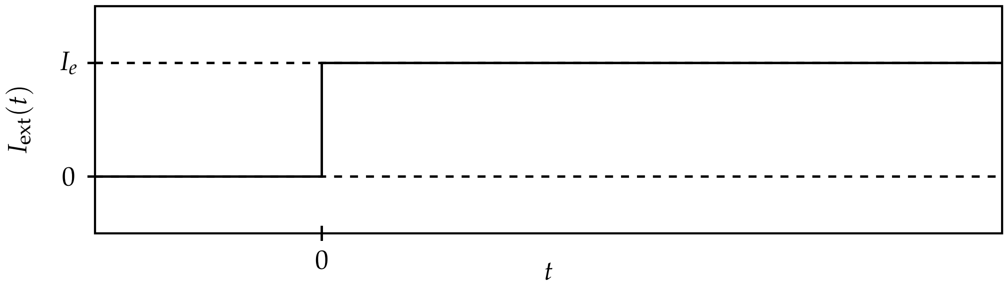


Figure 8: Simplified circuit model of a neuron membrane with an external current stimulus.

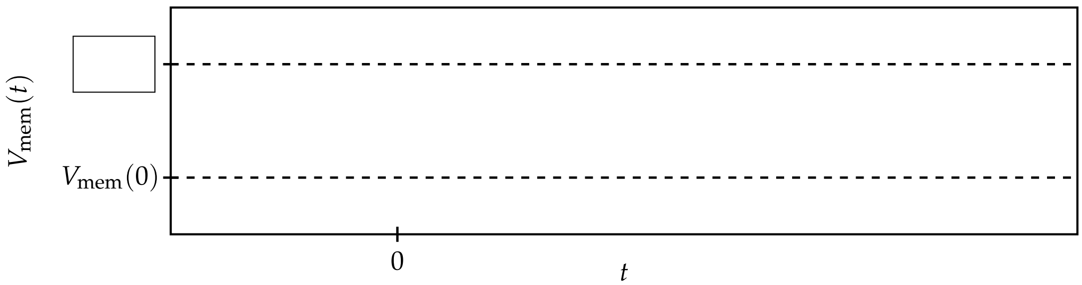
For all parts of this problem the external stimulus $I_{ext}(t)$ is a piece-wise constant function as shown below:



i. **Find the value of $V_{\text{mem}}(0)$ assuming the system reached steady-state for $t < 0$.**

ii. **Solve for $V_{\text{mem}}(t)$ where $t \geq 0$. Show your work.**

iii. **Qualitatively sketch $V_{\text{mem}}(t)$ on the below plot, and label the steady-state value by filling in the un-filled y-axis label.**



(b) As a part of his project, Krishna needs to measure the neural potentials. However, in the measurement process multiple neurons can come into contact. He came up with the following circuit modeling two neuron membranes in contact with each other.

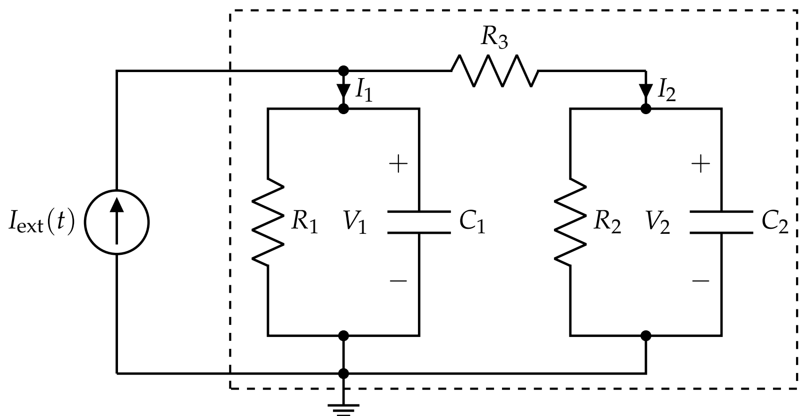


Figure 9: Simplified circuit model for two neuron membranes in contact.

By doing nodal analysis of the circuit, he found that the membrane voltages $V_1(t)$ and $V_2(t)$ are related to the external current stimulus ($I_{\text{ext}}(t)$) through the following vector differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} a & \frac{1}{R_3 C_1} \\ \frac{1}{R_3 C_2} & -\frac{1}{(R_2||R_3)C_2} \end{bmatrix} \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} I_{\text{ext}}(t) \tag{7}$$

where $R_i||R_j = \frac{R_i R_j}{R_i + R_j}$. **Find expressions for a and b in terms of R_1, R_2, R_3, C_1 and C_2.**

(c) Suppose for some appropriate component values, the vector differential equation 7 can be written in the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} -30 & 10 \\ 10 & -30 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 10^3 \\ 0 \end{bmatrix} u(t) \tag{8}$$

where $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix}$ and $u(t) = I_{\text{ext}}(t)$. The external current source, $I_{\text{ext}}(t)$ is same as in part (a) with $I_e = 10\,\text{mA}$. Plugging the value of $I_{\text{ext}}(t)$, for $t > 0$ eq. 8 becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} -30 & 10 \\ 10 & -30 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 10 \\ 0 \end{bmatrix} \tag{9}$$

Let's say the two eigenvalues of $\begin{bmatrix} -30 & 10 \\ 10 & -30 \end{bmatrix}$ are λ_1, λ_2 and the corresponding eigenvectors are $\vec{v}_1$, $\vec{v}_2$ respectively. Let's also define $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$. It's given to you that $\lambda_1 = -40$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. You can also consider $V^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

i. **Find the value of λ_2.**

ii. Let's define $\vec{\tilde{x}}(t)$ such that $\vec{x}(t) = V\vec{\tilde{x}}(t)$. **Determine** $\vec{\tilde{x}}(0)$. You can assume the initial condition of the circuit to be $\vec{x}(0) = \vec{0}$.

iii. **Now diagonalize the system given by eq. 9 and solve for $\vec{\tilde{x}}(t)$ for $t \geq 0$.**

iv. **Use the result in the previous part to find $\vec{x}(t)$ for $t \geq 0$.**

8. Active Filter

NOTE: This problem doesn't use any result from the previous problem on brain stimulation.

Krishna wants to measure the membrane potential of the neuron to characterize the neuron behavior as a part of his neural-modeling project. However, he knows that to accurately measure the neuron membrane potential, he needs to cancel out any external interference which may corrupt the neural signals. To do that he designed the following active filter:

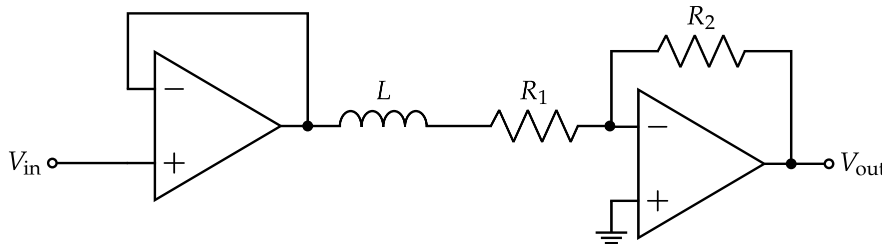


Figure 10: Schematic of the active filter used to cancel unwanted interference to the neuron membrane potential

(a) Assume the op-amps used in the filter are ideal. **Which of the following best describes the type of this filter?**

Filter type	Select one
2^{nd}-order low-pass filter	◯
1^{st}-order low-pass filter	◯
High-pass filter	◯
Band-pass filter	◯

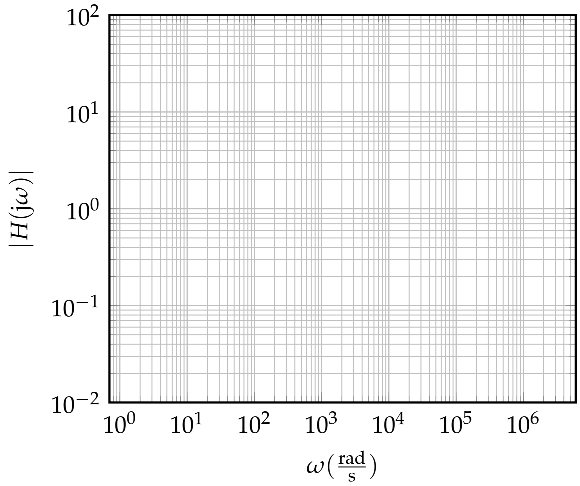
(b) **Derive the transfer function of the filter.**

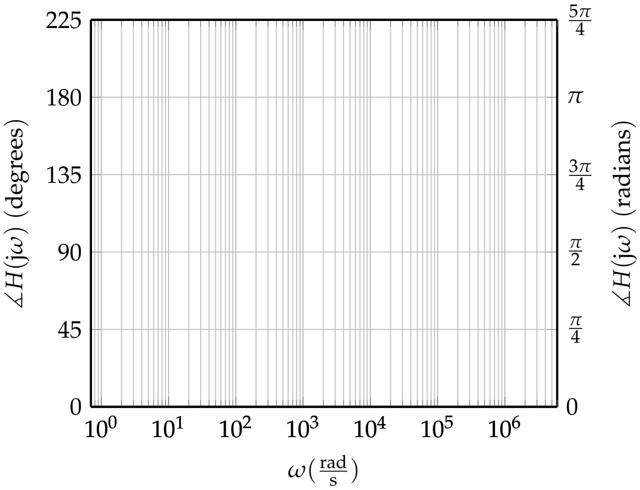
(c) Assuming $R_2 = 10 \times R_1$, **find the magnitude of the transfer function of the filter at** $\omega = 0$ **(i.e.** $|H(\mathrm{j} \cdot 0)|$**).**

(d) Now assume the frequency of the neural signal, ω_s can be in the range of 0 Hz to 100 Hz and the interference signal frequency, ω_{int} is 60 kHz. The filter cut-off frequency, ω_c needs to be positioned so that the interference is attenuated by at least a factor of 100 compared to $|H(\mathrm{j} \cdot 0)|$ (i.e. $|H(\mathrm{j}\omega_{\text{int}})| \leq \frac{|H(\mathrm{j} \cdot 0)|}{100}$) and the neural signal doesn't see any attenuation compared to $|H(\mathrm{j} \cdot 0)|$ (i.e. $|H(\mathrm{j}\omega_s)| \approx |H(\mathrm{j} \cdot 0)|$). **Which of the following is an acceptable range of cut-off frequencies (**ω_c**) for the active filter that Krishna designed?** *Justify your answer.*

Frequency range	Select one
50 Hz - 60 Hz	◯
500 Hz - 600 Hz	◯
5 kHz - 6 kHz	◯
50 kHz - 60 kHz	◯

(e) Suppose you have $R_1 = 10\,\Omega$, $R_2 = 100\,\Omega$ and $L = 10\,\text{mH}$. **Draw the Bode plot (straight-line approximations to the transfer function) for the magnitude and phase of the active filter.**





9. Affine Control

In this problem, we will analyze a *affine* model of the form

$$x[i+1] = \alpha x[i] + \beta u[i] + \gamma \tag{10}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$, $x\colon \mathbb{N} \to \mathbb{R}$ is the state, and $u\colon \mathbb{N} \to \mathbb{R}$ is the input. Affine models are ubiquitous in control theory – in fact, our robot car from lab obeys a two-state-variable affine model.

(a) Suppose (for this part only) that:

- $\alpha = 1$,
- $\beta = 0$,
- $\gamma \neq 0$,
- $x[0]$ is anything.

so the model is of the form

$$x[i+1] = x[i] + \gamma. \tag{11}$$

Is the state x bounded? *Justify your answer.*

(b) Suppose (for this part only) that the state evolves according to Equation (10), i.e.,

$$x[i+1] = \alpha x[i] + \beta u[i] + \gamma \tag{12}$$

and

- $\alpha \neq 0$,
- $\beta > 0$,
- $\gamma \neq 0$,
- $x[0] = 0$.

Suppose that we supply feedback control of the form

$$u[i] = f \cdot x[i] \tag{13}$$

for $f \in \mathbb{R}$.

i. **For the specific case of $f = \frac{-1-\alpha}{\beta}$, show that the state x is bounded.**

ii. **In terms of α and β, give a range of f that keeps the state x bounded.**

(c) Suppose (for this part only) that the state evolves according to Equation (10), i.e.,

$$x[i+1] = \alpha x[i] + \beta u[i] + \gamma \tag{14}$$

and

- α is anything,
- β is anything,
- γ is anything,
- $x[0]$ is anything.

Suppose that we are setting up a least-squares system identification procedure to learn α, β, and γ, and that we have data of the form $(x[i], u[i], x[i+1])$, for $i \in \{0, 1, \ldots, \ell - 1\}$. **Set up a least-squares problem $D\vec{p} \approx \vec{s}$ to learn estimates for α, β, γ. What are D, $\vec{p}$, and $\vec{s}$?**

NOTE: Your answer for D should be as compact as possible.

NOTE: You do not need to solve the least squares problem; just set it up.

(d) Suppose (for this part only) that the state evolves according to Equation (10), i.e.,

$$x[i+1] = \alpha x[i] + \beta u[i] + \gamma \tag{15}$$

and

- $\alpha > 1$,
- $\beta > 0$,
- $\gamma > 0$,
- $x[0]$ is anything.

Suppose that we actually got our discrete-time model

$$x[i+1] = \alpha x[i] + \beta u[i] + \gamma \tag{16}$$

by discretizing a continuous-time model

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = ax(t) + bu(t) + c \tag{17}$$

where the sampling interval length is $\Delta = 1$, i.e., $x[i] = x(i\Delta)$, and $u(t)$ is piecewise constant over intervals of length Δ, i.e., $u(t) = u(i\Delta) = u[i]$ for $t \in [i\Delta, (i+1)\Delta)$. **In terms of** α, β, γ**, what are** a, b**, and** c**?**

(HINT: You can use any discretization formulas we derived in class, as long as they apply. Alternatively, you may use the following formula in your derivation.
For a constant input v, and a time t_0 for which $x(t_0)$ is known, the solution to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = ax(t) + v \qquad t \geq t_0 \tag{18}$$

is given by

$$x(t) = \mathrm{e}^{a(t-t_0)}x(t_0) + \frac{\mathrm{e}^{a(t-t_0)} - 1}{a} \cdot v, \qquad t \geq t_0. \tag{19}$$

when $a \neq 0$. Also, recall from the problem statement above that the sampling interval length $\Delta = 1$.)