Midterm Redo

The midterm redo is due on Monday, March 28, 2022, at 11:59PM.

1. Complex Numbers

You are given the graph in Figure 1.

In this problem, you may use the $\text{atan2}(b, a)$ function to compute the angle (phase) for the complex number $a + jb$ as necessary.

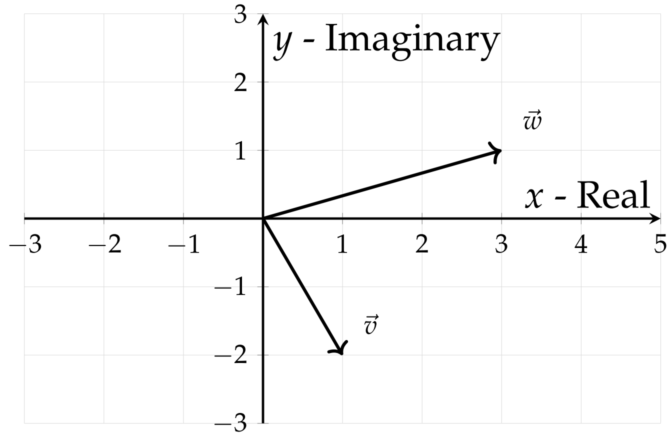


Figure 1: Vectors in the $\mathbf{x} - \mathbf{y}$ plane

(a) **What are the Cartesian ((x, y)) and Polar ($re^{j\theta}$) coordinates of $\vec{v}$?**

Guidance:

- Use the definition of Cartesian and polar coordinates from Note j.
- First try to find the Cartesian form, then convert that into polar using the formulas in the note.

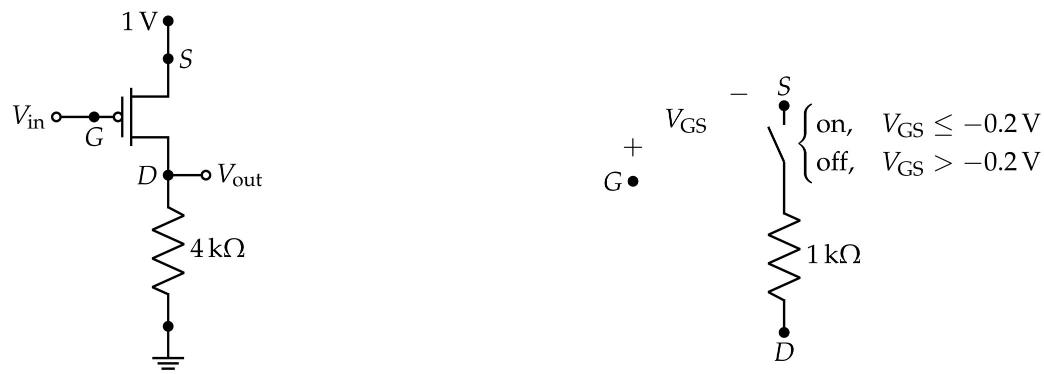
(b) **What are the Cartesian ((x, y)) and Polar ($re^{j\theta}$) coordinates of $\vec{w}$?**

Guidance:

- Use the definition of Cartesian and polar coordinates from Note j.
- First try to find the Cartesian form, then convert that into polar using the formulas in the note.

2. PMOS Transistor Inverter

Consider the following schematic and PMOS model.

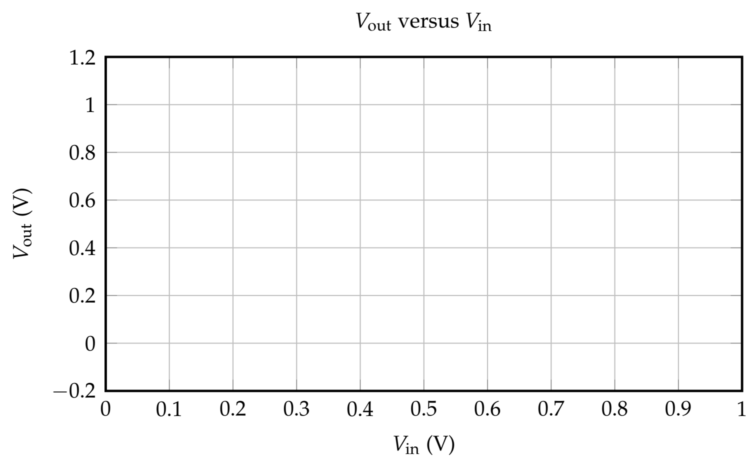


(a) A PMOS transistor circuit **(b)** Resistor and switch model for PMOS transistor.

Figure 2: PMOS figures.

Please plot the output V_{out} for the input V_{in} ranging from $0\,\text{V}$ **to** $1\,\text{V}$. *Justify your answer.*

NOTE: The y-axis ticks starts from $-0.2\,\text{V}$.



(I) V_{out} versus V_{in} Graph

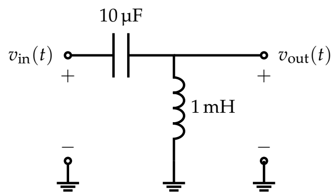
Guidance:

- First, substitute the transistor in the left figure with its switch model in the right figure.
- Use this to identify the value of V_{GS} at which the behavior of the circuit changes.
- Convert this value of V_{GS} into what the value of V_{in} is to produce this value of V_{GS}.
- Look at what happens when you take V_{in} higher and lower than this special value.
- Extrapolate to all V_{in} using the fact that V_{out} (as a function of V_{in}) is piecewise constant, with the two "pieces" being when V_{in} is before and after this special value.

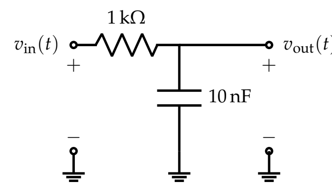
3. Filter Circuits

(a) Below, you have filter circuits A, B, C, D, each with specific component values. **Fill in the bubbles to match each filter to its corresponding magnitude transfer function plot out of choices I, II, III, IV.** Note that each plot may be assigned to filters **once, more than once, or not at all.** Each filter has **exactly one** corresponding plot.

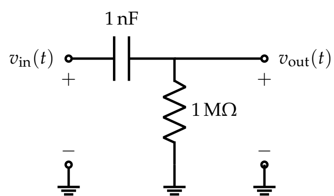
SI Prefixes and Exponent definitions: nano (n): 10^{-9}; micro (μ): 10^{-6}; milli (m): 10^{-3}; kilo (k): 10^{3}; mega (M): 10^{6}; giga (G): 10^{9}



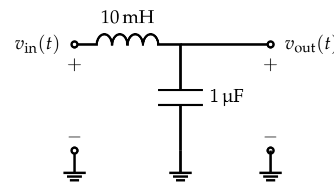
(A) Filter A.



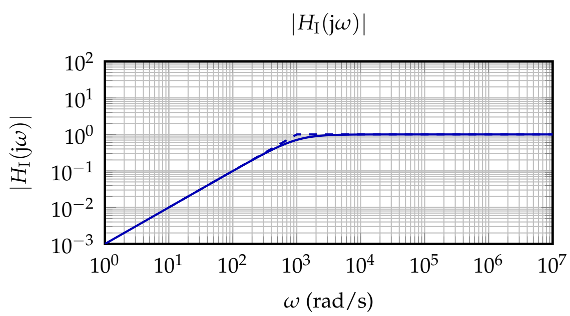
(B) Filter B.



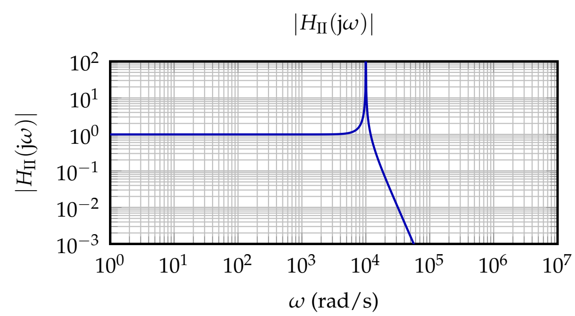
(C) Filter C.



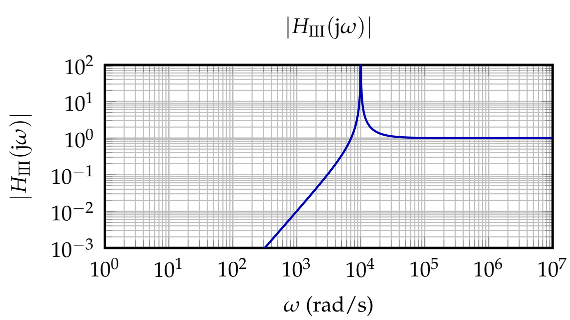
(D) Filter D.



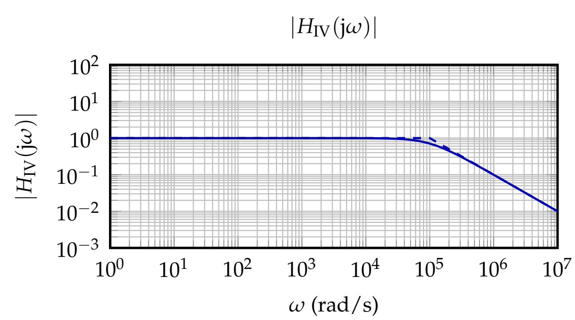
(I) Plot I.



(II) Plot II.



(III) Plot III.



(IV) Plot IV.

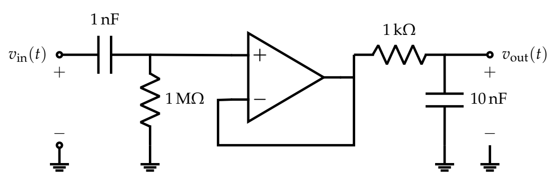
Filter Letter	Plot I	Plot II	Plot III	Plot IV
A	◯	◯	◯	◯
B	◯	◯	◯	◯
C	◯	◯	◯	◯
D	◯	◯	◯	◯

Guidance: For each circuit:

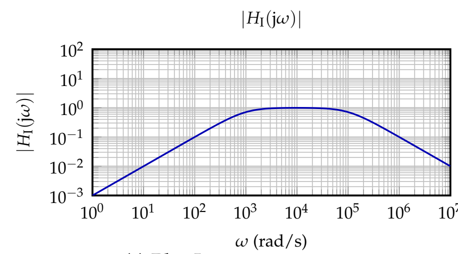
- Look at whether the circuit is LC or RC.
- Decide the "important" frequencies (for LC this is resonance, for RC this is cutoff frequency).
- For LC, what happens during resonance?
- If you still need to pick between two choices, find the transfer function and take limits $\omega \to 0$ and $\omega \to \infty$ to look at long-term behavior of the circuit (low-pass? at what gain? etc.)

(b) Now, in order to design a band-pass filter, one possible way is to cascade two filters above. Below, you have filter circuits A, B, C, each with specific component values. **Fill in the bubbles to match each filter to its corresponding magnitude transfer function plot out of choices I, II, III.**

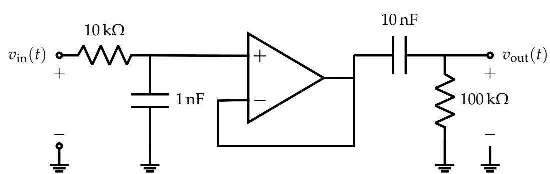
Note that each plot may be assigned to filters **once, more than once, or not at all.** Each filter has **exactly one** corresponding plot.



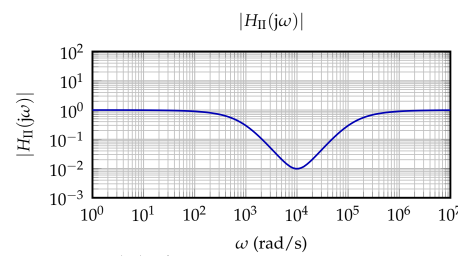
(A) Filter A.



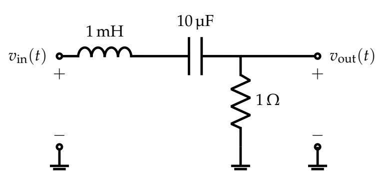
(I) Plot I.



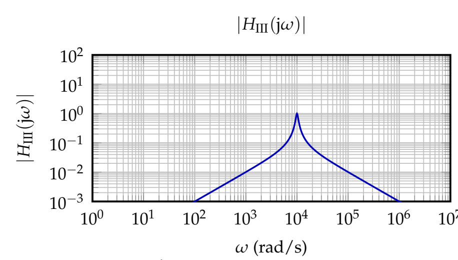
(B) Filter B.



(II) Plot II.



(C) Filter C.



III Plot III.

Filter Letter	Plot I	Plot II	Plot III
A	◯	◯	◯
B	◯	◯	◯
C	◯	◯	◯

Guidance: For each circuit:

- Find the transfer function. Recall how to find the transfer function of composed filters, which is just multiply the transfer functions of the component filters.
- Take limits $\omega \to 0$ and $\omega \to \infty$ of the transfer function to see the limiting magnitudes.
- If still picking between choices, note that RLC will have sharp peaks in the transfer function due to resonance effects, but RC and CR filters composed will never have resonance and so their transfer functions are smooth.

4. Magnitude, Phase, and Cascades

Suppose you have the transfer function $H(\mathrm{j}\omega)$ for a system as given in eq. (1) below:

$$H(\mathrm{j}\omega) = \frac{20}{1 + \mathrm{j}\frac{\omega}{\omega_0}} \tag{1}$$

where $\omega_0 := 1 \times 10^7 \, \frac{\text{rad}}{\text{s}}$.

Answer the following questions.

(a) **What is the transfer function's magnitude $|H(\mathrm{j}\omega)|$ at $\omega = 0 \, \frac{\text{rad}}{\text{s}}$?**

Guidance:

- Compute the magnitude of the transfer function. Note that $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.
- Plug in $\omega = 0$ and evaluate.

(b) **What is the transfer function's magnitude $|H(\mathrm{j}\omega)|$ at $\omega = \infty \frac{\text{rad}}{\text{s}}$?**

Guidance:

- Compute the magnitude of the transfer function if you don't have it already.
- Take the limit $\omega \to \infty$ of the magnitude of the transfer function.

(c) **What is the transfer function's phase $\measuredangle H(\mathrm{j}\omega)$ at $\omega = 1 \times 10^7 \, \frac{\text{rad}}{\text{s}}$?**

Guidance:

- Compute the phase of the transfer function. Note that $\measuredangle \frac{a}{b} = \measuredangle a - \measuredangle b$.
- Plug in $\omega = 1 \times 10^7 \, \frac{\text{rad}}{\text{s}}$ and evaluate.

(d) You cascade the systems S_1 as defined by the transfer function:

$$H_1(\mathrm{j}\omega) = \frac{20}{1 + \mathrm{j}\frac{\omega}{\omega_1}} \tag{2}$$

with another system S_2 as defined by the transfer function:

$$H_2(\mathrm{j}\omega) = \frac{100}{1 + \mathrm{j}\frac{\omega}{\omega_2}}. \tag{3}$$

where $\omega_1 := 1 \times 10^7 \, \frac{\text{rad}}{\text{s}}$ and $\omega_2 := 1 \times 10^4 \, \frac{\text{rad}}{\text{s}}$.

You place S_2 after S_1, with unity-gain buffers in between. **Write the overall transfer function $H_{\text{cascade}}(\mathrm{j}\omega)$ in terms of $\mathrm{j}\omega$.** *You do not need to simplify your answer for this subpart.*

Guidance:

- Recall that the overall transfer function of a composed system is the product of the component transfer functions.

5. Stability of Discrete-Time System

Suppose we are working with a linear model which has the form:

$$\underbrace{\begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix}}_{\vec{x}[i+1]} = \underbrace{\begin{bmatrix} 0 & 1 \\ k & 1 \end{bmatrix}}_{A_d} \underbrace{\begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix}}_{\vec{x}[i]} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w[i], \tag{4}$$

where $k \in \mathbb{R}$ is an unknown variable.

(a) **Give the range of k, such that the matrix A_d has only real eigenvalues**. *Justify your answer.*

Guidance:

- Compute the characteristic polynomial of A_d. You should get a quadratic function.
- Find the eigenvalues of the characteristic polynomial symbolically. You should get a pair of eigenvalues. A square root of a simple function of k, say $\sqrt{f(k)}$, will be involved.
- Find the range of k such that the the function of k, i.e., $f(k)$, is non-negative.

(b) **Choose the possible k value(s) from the following options such that the above model is stable.** *Select all choices that apply.*

i. $k = -\frac{3}{4}$.
ii. $k = 1$.
iii. $k = -\frac{1}{4}$.
iv. $k = \frac{1}{4}$.

Guidance:

- Recall that the condition for discrete-time stability is that all eigenvalues have magnitude < 1.
- Using the eigenvalues polynomial you computed in the last part, plug in the different values of k that are given, and find the magnitudes of both eigenvalues for this k; then you can check their magnitudes.

6. Controllability and Eigenvalue Placement

Suppose we are working with a linear model with two-dimensional state $\vec{x}\colon \mathbb{N} \to \mathbb{R}^2$ but one-dimensional input $u\colon \mathbb{N} \to \mathbb{R}$:

$$\underbrace{\begin{bmatrix} x_1[i+1] \\ x_2[i+1] \end{bmatrix}}_{=\vec{x}[i+1]} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{=A} \underbrace{\begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix}}_{=\vec{x}[i]} + \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{=\vec{b}} u[i] \tag{5}$$

where $b_1 \neq 0$ and $b_2 \neq 0$.

(a) **Show that the model in Equation (5) is not controllable.**

Guidance:

- Recall the definition of the controllability matrix: $\mathcal{C} = \begin{bmatrix} A\vec{b} & \vec{b} \end{bmatrix}$.
- Note that $A = I_2$ the identity matrix. Simplify the controllability matrix using this.
- Finish off by applying the definition of controllability.

(b) Suppose we add feedback control of the form

$$u[i] := \underbrace{\begin{bmatrix} f_1 & f_2 \end{bmatrix}}_{=\vec{f}^\top} \underbrace{\begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix}}_{=\vec{x}[i]}. \tag{6}$$

Show that one of the eigenvalues of $A_{\mathrm{CL}} := A + \vec{b}\vec{f}^\top$ is 1, regardless of the values of f_1 and f_2.

Guidance:

- Write out all entries of $A_{\mathrm{CL}} - \lambda I$ in terms of entries of $A, \vec{b}, \vec{f}, \lambda$.
- Compute determinant of $A_{\mathrm{CL}} - \lambda I$ (characteristic polynomial of A_{CL})
- Plug in $\lambda = 1$ and show the determinant of $A_{\mathrm{CL}} - \lambda I$ is 0. This means 0 is an eigenvalue of A_{CL}.

7. Brain Stimulation

For his neuron-modeling project Krishna thought of consulting his close friend Radhika, who is a neuroscientist. According to Radhika's suggestions, Krishna came up with the following model of the cell-membrane of a neuron:

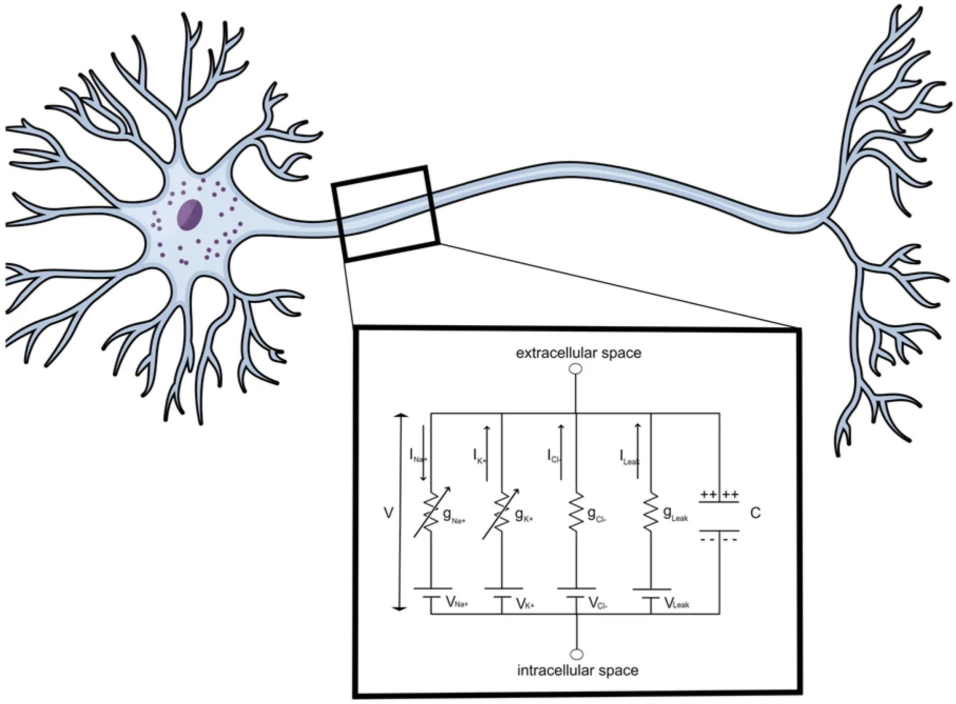


Figure 7: Electrical model of the neuron membrane

(a) Now Krishna wants to see how the neuron behaves to an external current stimulus. As he found the complete model very difficult to analyze, he starts his analysis with the following simple model:

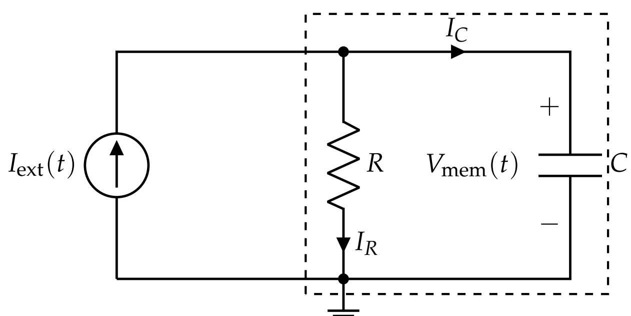
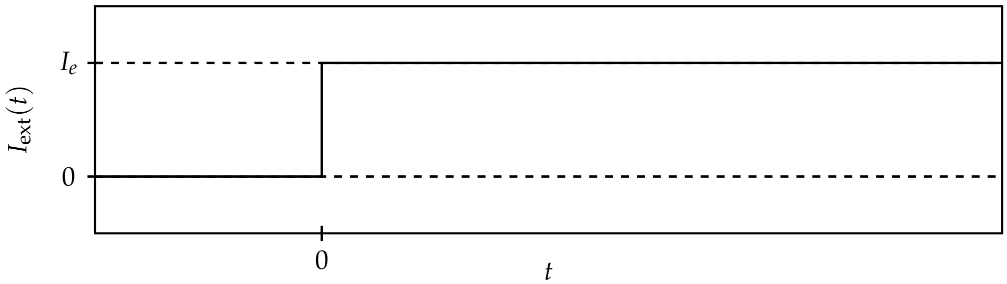


Figure 8: Simplified circuit model of a neuron membrane with an external current stimulus.

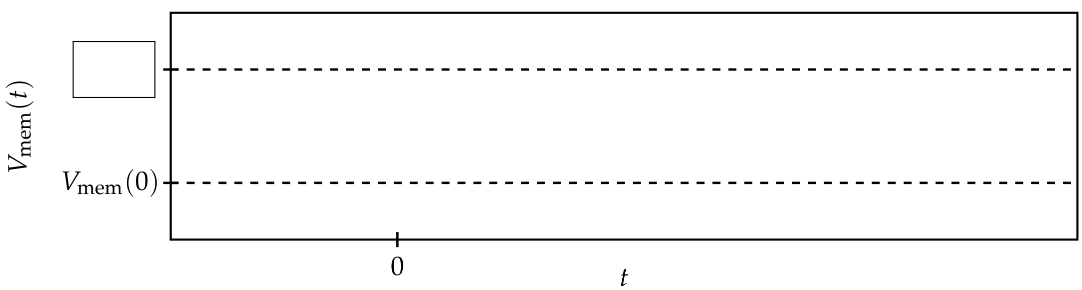
For all parts of this problem the external stimulus $I_{\text{ext}}(t)$ is a piece-wise constant function as shown below:



i. **Find the value of $V_{mem}(0)$ assuming the system reached steady-state for $t < 0$.**
Guidance: Note that $I_{ext}(t) = 0$ for $t < 0$. What are steady-state values for this configuration (i.e., $t < 0$)?

ii. **Solve for $V_{mem}(t)$ where $t \geq 0$. Show your work.**
Guidance:
- Start off with KCL.
- Use I-V relationships to get differential equation for $\frac{d}{dt}V_{mem}(t)$ in terms of $V_{mem}(t)$ and some constants.
- Plug in $I_{ext}(t) = I_e$ if you haven't done that already.
- Solve the scalar differential equation using the initial condition from part (a). You can use i.e., Note 2 Section 4.5, as a help.

iii. **Qualitatively sketch $V_{mem}(t)$ on the below plot, and label the steady-state value by filling in the un-filled y-axis label.**



Guidance:
- Until time $t = 0$, what is $V_{mem}(t)$? (Recall that we are in steady-state before $t = 0$).
- For $t \geq 0$, plot $V_{mem}(t)$ using our expression from the previous part.

(b) As a part of his project, Krishna needs to measure the neural potentials. However, in the measurement process multiple neurons can come into contact. He came up with the following circuit modeling two neuron membranes in contact with each other.

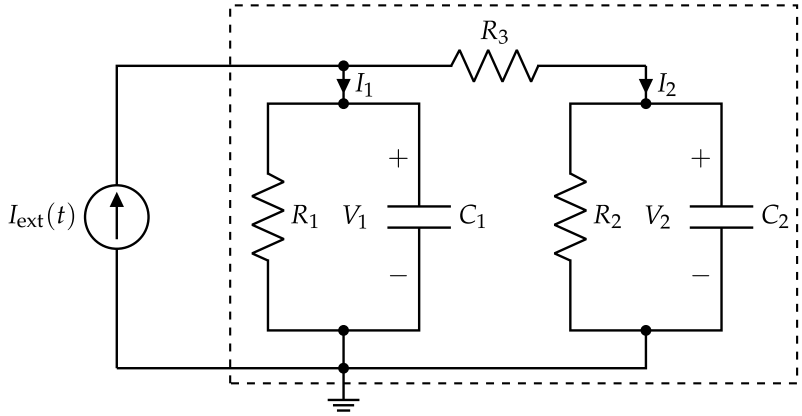


Figure 9: Simplified circuit model for two neuron membranes in contact.

By doing nodal analysis of the circuit, he found that the membrane voltages $V_1(t)$ and $V_2(t)$ are related to the external current stimulus ($I_{\text{ext}}(t)$) through the following vector differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} = \begin{bmatrix} a & \frac{1}{R_3C_1} \\ \frac{1}{R_3C_2} & -\frac{1}{(R_2||R_3)C_2} \end{bmatrix}\begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} I_{\text{ext}}(t) \tag{7}$$

where $R_i||R_j = \frac{R_iR_j}{R_i+R_j}$. **Find expressions for a and b in terms of R_1, R_2, R_3, C_1 and C_2.**

Guidance:

- Apply KCL and I-V relationships to get a system for $\frac{\mathrm{d}}{\mathrm{d}t}V_1(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}V_2(t)$ in terms of $I_1(t)$, $I_2(t)$, $V_1(t)$, $V_2(t)$, and some constants.
- Use Ohm's law on I_2 to write it in terms of $V_1(t)$, $V_2(t)$, and some constants.
- Use KCL to find a relation between $I_1(t)$, $I_2(t)$, and $I_{\text{ext}}(t)$.
- Use the previous two relations to write $I_1(t)$ in terms of $I_{\text{ext}}(t)$, $V_1(t)$, $V_2(t)$, and some constants.
- Plug these expressions for $I_1(t)$, $I_2(t)$ into our original system we found using KCL.
- Simplify.

(c) Suppose for some appropriate component values, the vector differential equation 7 can be written in the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} -30 & 10 \\ 10 & -30 \end{bmatrix}\vec{x}(t) + \begin{bmatrix} 10^3 \\ 0 \end{bmatrix} u(t) \tag{8}$$

where $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix}$ and $u(t) = I_{\text{ext}}(t)$. The external current source, $I_{\text{ext}}(t)$ is same as in part (a) with $I_e = 10\,\text{mA}$. Plugging the value of $I_{\text{ext}}(t)$, for $t > 0$ eq. 8 becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x}(t) = \begin{bmatrix} -30 & 10 \\ 10 & -30 \end{bmatrix}\vec{x}(t) + \begin{bmatrix} 10 \\ 0 \end{bmatrix} \tag{9}$$

Let's say the two eigenvalues of $\begin{bmatrix} -30 & 10 \\ 10 & -30 \end{bmatrix}$ are λ_1, λ_2 and the corresponding eigenvectors are $\vec{v}_1, \vec{v}_2$ respectively. Let's also define $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$. It's given to you that $\lambda_1 = -40$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. You can also consider $V^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

i. **Find the value of λ_2.**
Guidance:

- Since you already know $\vec{v}_2$, compute $A\vec{v}_2$ (for $A = \begin{bmatrix} -30 & 10 \\ 10 & -30 \end{bmatrix}$).
- Identify λ_2 such that $A\vec{v}_2 = \lambda_2\vec{v}_2$.

ii. Let's define $\vec{\tilde{x}}(t)$ such that $\vec{x}(t) = V\vec{\tilde{x}}(t)$. **Determine** $\vec{\tilde{x}}(0)$. You can assume the initial condition of the circuit to be $\vec{x}(0) = \vec{0}$.

Guidance: We already know $\vec{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. What is $V^{-1}\vec{x}(0)$?

iii. **Now diagonalize the system given by eq. 9 and solve for $\vec{\tilde{x}}(t)$ for $t \geq 0$.**
Guidance:

- Write the original system in $\vec{\tilde{x}}$ coordinates. See Note 3 if this is giving you trouble.
- Once in diagonalized coordinates, solve each row of the diagonal system for $\tilde{x}_k(t)$.

iv. **Use the result in the previous part to find $\vec{x}(t)$ for $t \geq 0$.**
Guidance: We have the system solved in the diagonalized coordinates, so it just remains to do a change-of-basis into the regular coordinates (basically multiplying by V).

8. Active Filter

NOTE: This problem doesn't use any result from the previous problem on brain stimulation.

Krishna wants to measure the membrane potential of the neuron to characterize the neuron behavior as a part of his neural-modeling project. However, he knows that to accurately measure the neuron membrane potential, he needs to cancel out any external interference which may corrupt the neural signals. To do that he designed the following active filter:

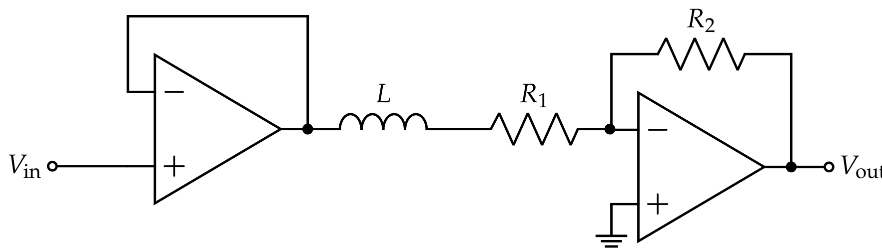


Figure 10: Schematic of the active filter used to cancel unwanted interference to the neuron membrane potential

(a) Assume the op-amps used in the filter are ideal. **Which of the following best describes the type of this filter?**

Filter type	Select one
2^{nd}-order low-pass filter	◯
1^{st}-order low-pass filter	◯
High-pass filter	◯
Band-pass filter	◯

Guidance:

- It is probably easiest to find the transfer function.
- Note that the right-hand side of the circuit looks like an inverting amplifier.
- Working in frequency domain with impedances, all the op-amp configurations still work (e.g. inverting amplifier), but with impedances and voltage phasors instead of resistances and voltages.
- Inductor's and resistor's impedances are in series, so find their equivalent impedance by adding them.
- Then we exactly get an inverting amplifier in phasor domain.
- To find the transfer function, we can just use our knowledge of inverting amplifier to conclude that it's negative of the impedance ratio.

(b) **Derive the transfer function of the filter.**
Guidance: Same hints as above.

(c) Assuming $R_2 = 10 \times R_1$, **find the magnitude of the transfer function of the filter at $\omega = 0$ (i.e. $|H(\mathrm{j} \cdot 0)|$).**
Guidance: Plug in $\omega = 0$ into the transfer function you found, and evaluate.

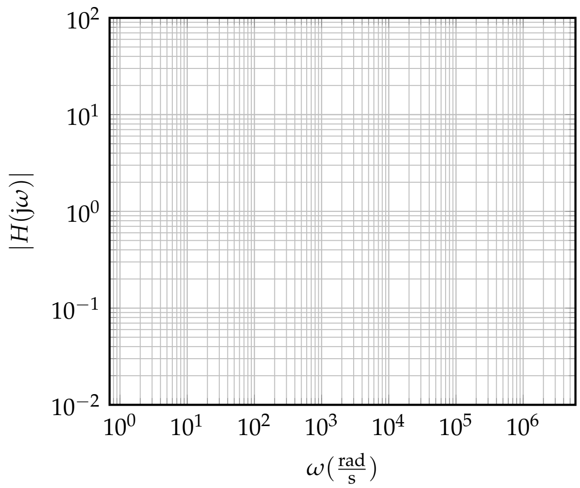
(d) Now assume the frequency of the neural signal, ω_s can be in the range of 0 Hz to 100 Hz and the interference signal frequency, ω_{int} is 60 kHz. The filter cut-off frequency, ω_c needs to be positioned so that the interference is attenuated by at least a factor of 100 compared to $|H(\mathrm{j} \cdot 0)|$ (i.e. $|H(\mathrm{j}\omega_{\text{int}})| \leq \frac{|H(\mathrm{j}\cdot 0)|}{100}$) and the neural signal doesn't see any attenuation compared to $|H(\mathrm{j} \cdot 0)|$ (i.e. $|H(\mathrm{j}\omega_s)| \approx |H(\mathrm{j} \cdot 0)|$). **Which of the following is an acceptable range of cut-off frequencies (ω_c) for the active filter that Krishna designed?** *Justify your answer.*

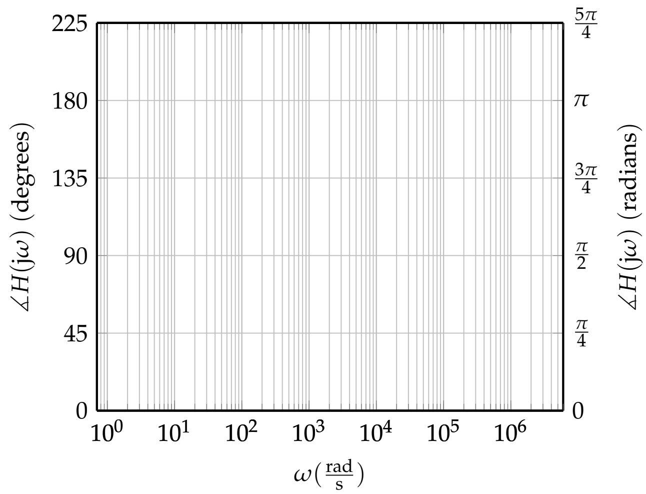
Frequency range	Select one
50 Hz - 60 Hz	◯
500 Hz - 600 Hz	◯
5 kHz - 6 kHz	◯
50 kHz - 60 kHz	◯

Guidance:

- We want to preserve 0 Hz to 100 Hz while rejecting 60 kHz, attenuating it by a factor of 100.
- This means that the cutoff frequency should be 100 times lower than 60 kHz.

(e) Suppose you have $R_1 = 10\,\Omega$, $R_2 = 100\,\Omega$ and $L = 10\,\text{mH}$. **Draw the Bode plot (straight-line approximations to the transfer function) for the magnitude and phase of the active filter.**





Guidance: Refer to Note 8 for a how-to on making Bode plots you know the transfer function of.

9. Affine Control

In this problem, we will analyze a *affine* model of the form

$$x[i+1] = \alpha x[i] + \beta u[i] + \gamma \tag{10}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$, $x\colon \mathbb{N} \to \mathbb{R}$ is the state, and $u\colon \mathbb{N} \to \mathbb{R}$ is the input. Affine models are ubiquitous in control theory – in fact, our robot car from lab obeys a two-state-variable affine model.

(a) Suppose (for this part only) that:

- $\alpha = 1$,
- $\beta = 0$,
- $\gamma \neq 0$,
- $x[0]$ is anything.

so the model is of the form

$$x[i+1] = x[i] + \gamma. \tag{11}$$

Is the state x bounded? *Justify your answer.*

Guidance:

- Write out the first few terms $x[0], x[1], x[2], \ldots$.
- Try to find a general formula for $x[i]$.
- Take the limit as $i \to \infty$.

(b) Suppose (for this part only) that the state evolves according to Equation (10), i.e.,

$$x[i+1] = \alpha x[i] + \beta u[i] + \gamma \tag{12}$$

and

- $\alpha \neq 0$,
- $\beta > 0$,
- $\gamma \neq 0$,
- $x[0] = 0$.

Suppose that we supply feedback control of the form

$$u[i] = f \cdot x[i] \tag{13}$$

for $f \in \mathbb{R}$.

i. **For the specific case of $f = \frac{-1-\alpha}{\beta}$, show that the state x is bounded.**

Guidance:

- Write out the first few terms $x[0], x[1], x[2], \ldots$.
- Try to find a general formula for $x[i]$.
- Find an upper bound for $|x[i]|$.

ii. **In terms of α and β, give a range of f that keeps the state x bounded.**

Guidance:

- Use the stability criteria $|\lambda| < 1$ to produce two inequalities for $\alpha + \beta f$.
- Notice that $\alpha + \beta f = -1$ is allowed (in this case) by (i) (why? show this if you don't feel comfortable with it).
- Notice that $\lambda = \alpha + \beta f = 1$ is not allowed (in this case) by part (a) (again, why? show this if you don't feel comfortable with it).
- Modify your inequalities for $\alpha + \beta f$ accordingly.
- Turn those inequalities into a range for f.

(c) Suppose (for this part only) that the state evolves according to Equation (10), i.e.,

$$x[i+1] = \alpha x[i] + \beta u[i] + \gamma \tag{14}$$

and

- α is anything,
- β is anything,
- γ is anything,
- $x[0]$ is anything.

Suppose that we are setting up a least-squares system identification procedure to learn α, β, and γ, and that we have data of the form $(x[i], u[i], x[i+1])$, for $i \in \{0, 1, \ldots, \ell - 1\}$. **Set up a least-squares problem $D\vec{p} \approx \vec{s}$ to learn estimates for α, β, γ. What are D, $\vec{p}$, and $\vec{s}$?**

NOTE: Your answer for D should be as compact as possible.

NOTE: You do not need to solve the least squares problem; just set it up.

Guidance: Follow the method in Note 10, except you will need to add a column of 1s to account for γ.

(d) Suppose (for this part only) that the state evolves according to Equation (10), i.e.,

$$x[i+1] = \alpha x[i] + \beta u[i] + \gamma \tag{15}$$

and

- $\alpha > 1$,
- $\beta > 0$,
- $\gamma > 0$,
- $x[0]$ is anything.

Suppose that we actually got our discrete-time model

$$x[i+1] = \alpha x[i] + \beta u[i] + \gamma \tag{16}$$

by discretizing a continuous-time model

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = a x(t) + b u(t) + c \tag{17}$$

where the sampling interval length is $\Delta = 1$, i.e., $x[i] = x(i\Delta)$, and $u(t)$ is piecewise constant over intervals of length Δ, i.e., $u(t) = u(i\Delta) = u[i]$ for $t \in [i\Delta, (i+1)\Delta)$. **In terms of α, β, γ, what are a, b, and c?**

(HINT: You can use any discretization formulas we derived in class, as long as they apply. Alternatively, you may use the following formula in your derivation.

For a constant input v, and a time t_0 for which $x(t_0)$ is known, the solution to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = a x(t) + v \qquad t \geq t_0 \tag{18}$$

is given by

$$x(t) = \mathrm{e}^{a(t-t_0)} x(t_0) + \frac{\mathrm{e}^{a(t-t_0)} - 1}{a} \cdot v, \qquad t \geq t_0. \tag{19}$$

when $a \neq 0$. Also, recall from the problem statement above that the sampling interval length $\Delta = 1$.)

Guidance:

- Define $v(t) = bu(t) + c$, so that $v[i] = bu[i] + c$. (This makes your affine system into a linear system, just with a different input). Note that since u is piecewise constant, v is also piecewise constant.
- Get an expression for $x[i+1]$ in terms of $x[i]$. There are two ways you can go about this:
  - Use the hint and plug in $t_0 = i\Delta$, $t = (i+1)\Delta$ since we want to find values at $(i+1)\Delta$ given the values at $i\Delta$, and then $v = v[i]$ (which works since v is piecewise constant).
  - Read off the discretization coefficients from Note 10.
- Read off α, β, γ as the coefficient of $x[i]$, the coefficient of $u[i]$, and the constant term, respectively, in the expression for $x[i+1]$ given $x[i]$.
- Solve for a, b, c from α, β, γ. Solve for a first and use this to simplify the other expressions.